MODERN ALGEBRA
THIRD SEMESTER COURSE

WELLS AND HART

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PREFACE

This text supplies subject matter and instruction for the *third semester course in algebra.* A clearly indicated basic or minimum course is offered for those schools which require or desire a brief course; equally clearly indicated optional topics extend this basic course for schools which can do more than a minimum course.

Special efforts have been made to prevent the first seven chapters from becoming a mere "re-do" of the first course in algebra. **Diagnostic Tests** covering subject matter about which the pupils know something from their first course appear in these chapters to help both the pupils and the teacher discover what parts of these chapters need special attention. They can serve not only as a means of diagnosis but as a challenge to the pupils; they are designed to stimulate the pupils to a desire for self-diagnosis, and to a sense of responsibility for self-improvement. Of course, any teacher who does not wish to use them for these purposes can omit them altogether or can use them as additional practice material or as pre-tests toward the end of the chapter in preparation for subsequent tests to be given by the teacher.

The Diagonostic Tests are followed by **remedial instruction and practice** of a relatively simple sort. These parts of the seven chapters have been extended a bit more than in preceding texts for this grade by the same authors to compensate for the simplification that has taken place in the first course in algebra in recent years. Pupils and schools not needing this elementary practice should omit it by all means. For such there are included more stimulating examples in lists having the suffix "b." (See Ex. 4, b, p. 10.) Besides, there are in these chapters *clearly marked new topics* (See pp. 21, 32, 39, 88, 110) some of which are a part of the minimum course, and some of which are optional. (See pp. 21, 23, 28, 52, 57, etc.) By these means these chapters are placed on a higher plane than corresponding chapters in the first course.

Attention is called to the chapter on **Functional Relationship.** (See p. 59.) The desire to place in the hands of teachers and pupils a satisfactory treatment of this subject, which has come to be stressed in recent years, was one of the chief reasons for writing this new text. The treatment will be found simple and adequate without being verbose or extended unnecessarily.

The **basic or minimum course** includes, first, as much of the remedial instruction and practice as the class needs; second, those topics which are not marked "optional"; and third, those exercises and problems which are not accompanied by the suffix "b" or which do not accompany an optional topic. This basic course includes all that most colleges can possibly require from candidates for admission. Whether or not a particular school or individual pupils should be expected to master this minimum course must be determined by local conditions.

The **optional topics** and more difficult examples are included as part of the long established policy of Wells and Hart texts of providing materials for the able pupils and classes beyond the mere basic requirements. To neglect such pupils is just as great a fault educationally as it is to overtax the less capable ones. Only by some such means can the full fruits of segregation of pupils into "X, Y, and Z" sections be secured and the widely discussed and generally admitted needs of pupils having "differences in ability" be provided for. In a *Handbook for Teachers* which accompanies this text, the author offers suggestions about means of utilizing such optional material, gathered from his own extensive experience.

Chapter VII gives only the subject of square root and quadratic surds, as in forerunners of this text. The subject of cube root, and other radicals, and the formal operations with them and exponents have no significance as preparation for the chapter on quadratic equations, and little significance outside of college entrance requirements. Such parts of radicals and exponents as are required for admission to certain colleges appear in Chapter XII; this chapter can be taught immediately after Chapter VII if the teacher wishes.

Chapter VIII includes a **complete treatment of Quadratic Equations.** This chapter presents an innovation in that the Theory of Quadratics is included in it instead of being left for a remote chapter. In particular, part of this "theory" is made to function in the course by being used as a means of checking the solution of a quadratic. (See foot of pp. 120, 123, and p. 127.) Attention is directed also to the means of motivating this whole chapter which appears on pages 114 and 115.

Graphs do not appear as a separate chapter in this text. They are made to function as a vital part of the teaching procedure, being used as a most valuable and vivid means of illuminating otherwise abstract subject matter. (See pp. 60, 65, 76, 78, 115, 116, 118, etc.) So used, graphs are an important part of *really fused mathematics* — not mere general mathematics.

Attention has been given to modern views about *accuracy in computation.* (See p. 68.) In the chapters on trigonometry and logarithms these views are used, as well as in the computations introduced under the subject of the formula.

The chapter on **trigonometry** gives complete instruction in that part of the subject which is now a part of the first course in algebra, and, besides, it includes the solution by logarithms of the same kind of problems, for this is now a part of the requirements in third semester algebra for certain institutions.

As parts of **a systematic teaching procedure**, this text provides full instruction on each topic, an adequate amount of practice exercises, additional exercises at the back of the text, cumulative reviews, and chapter mastery-tests. (See for these last the close of each of the chapters after the first few.) In so doing this text recognizes the progress made in the theory of teaching and furnishes teachers the means of applying this valuable theory in this particular subject.

In *style* and *workmanship* this text will be found a fit sequel to its predecessor, the Wells and Hart Revised Modern First Year Algebra.

CONTENTS

ALGEBRA

THIRD SEMESTER

I. THE FUNDAMENTAL OPERATIONS

1. How much elementary algebra do you still know? Can you make 100% on the following test?

DIAGNOSTIC TEST 1

Signed Numbers and Algebraic Expressions

How much is:

1. $(+5) + (+3)$?
2. $(+9) + (-3)$?
3. $(-10) + (+4)$?
4. $(-6) + (-4)$?
5. $(+6) - (+3)$?
6. $(-7) - (-2)$?
7. $(+8) - (-4)$?
8. $(-5) - (+7)$?
9. $(+7) \times (+3)$?
10. $(-5) \times (-4)$?
11. $(+6) \times (-2)$?
12. $(-7) \times (+4)$?
13. $(+12) \div (+2)$?
14. $(-9) \div (-3)$?
15. $(+6) \div (-2)$?
16. $(-10) \div (+2)$?
17. $5^2 = ?$
18. $4^3 = ?$
19. $2^4 = ?$
20. $(-2)^3 = ?$
21. $3x^2y$ is called a ____________.
22. $a + b$ is called a ______________.
23. $x + y - z$ is called a ____________.
24. x, y, and z are called ____________ of $x + y - z$.
25. In $2x^3y^4$, 2 is the ______________; 3 is the _____________ of x, and 4 is the _____________ of y.
26. The numerical value of $3x$ depends upon the _______ ____________ __________. When x increases, then _______ ____________.

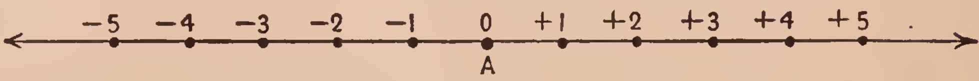
NOTE. Now read paragraphs 2 and 3 to learn how to answer any questions which you missed. If you made less than 100%, do all of Remedial Practice 1.

2. Positive and negative numbers are used to designate *oppositeness*. Thus, if $+5$ steps means five steps to the right, -5 steps means five steps to the left.

(*a*) The **absolute value** of a positive or a negative number is the arithmetical value remaining when the sign of the given number is omitted.

Thus, the absolute value of -3 or of $+3$ is 3.

(*b*) The points on a line are often indicated thus:



(*c*) Just as $5 *gain* combined with $5 *loss* produces 0,

so $$(+3) + (-3) = 0; \quad (+n) + (-n) = 0.$$

(*d*) Just as $5 *gain* combined with $3 *gain* gives $8 gain, so $(+5) + (+3) = +8$. Similarly $(-5) + (-3) = -8$.

To add two numbers having the same sign, *add their absolute values and prefix their common sign.*

(*e*) Just as $5 *gain* combined with $3 *loss* gives $2 gain,

so $$(+5) + (-3) = +2; \quad (-15) + (+5) = -10.$$

To add two numbers having unlike signs, *subtract the smaller absolute value from the larger and prefix to the result the sign of the number having the larger absolute value.*

(*f*) Since $(-3) + (-7) = -10$, then $(-10) - (-3) = -7$. This result can be secured by adding $+3$ to -10; that is, $(-10) + (+3) = -7$. This suggests the rule:

To subtract one number from another, *change the sign of the subtrahend and add the result to the minuend.*

Thus: $$(-6) - (+4) = (-6) + (-4), \text{ or } -10.$$

The **subtrahend** is the number subtracted and the **minuend** is the number from which the subtrahend is subtracted.

(*g*) Adding a negative number gives the same result as subtracting the corresponding positive number.

Thus: $(+11) + (-3)$ gives the same result as $(+11) - (+3)$.

(*h*) **For multiplication**, the following *definition* is used: **To multiply two numbers,** *multiply their absolute values; make the product positive if the numbers have like signs and negative if they have unlike signs.*

Thus: $(-6) \times (-7) = +42$; and $(-8) \times (+3) = -24$.

(*i*) **To divide one number by another,** *find the quotient of their absolute values; make it positive if they have like signs, and negative if they have unlike signs.*

Thus: $(-24) \div (+6) = -4$; and $(-32) \div (-16) = +2$.

The **quotient** of two numbers means the first divided by the second.

3. An **algebraic expression** is a number expressed by literal and arithmetical numbers.

The *arithmetical value* of an algebraic expression depends on the numerical value (or values) of the literal number (or numbers) in it.

EXERCISE 1. REMEDIAL PRACTICE

1. Give the *sum*, the *difference*, the *product*, and the *quotient* of each of the following pairs of numbers:

a. $+6$, and $+2$ *d.* -12, and -4 *g.* $+8$, and -24
b. -18, and $+3$ *e.* $+32$, and -4 *h.* -7, and $+14$
c. $+15$, and -5 *f.* -36, and $+6$ *i.* -5, and -3

2. What is the value of 4^3? of 2^4? of $(-3)^2$? of $(-2)^3$?

3. What is the value of $3x^2yz^3$ when $x = -2$, $y = 3$, and $z = -1$?

4. What is the value of each of the following expressions for the values of x, y, and z given in Example 3?

a. $5x^4$ *c.* $x^2 + y^2$ *e.* $x^2 - 2xy + y^2$ *g.* $xy + yx + xz$
b. x^3y *d.* $x^3 - y^3$ *f.* $z^2 - 5z - 8$ *h.* $x^3 + y^3 + z^3$

5. How much is: *a.* $(-1)^3$? *b.* $(-1)^6$? *c.* $(-1)^4$?

DIAGNOSTIC TEST 2

Addition and Subtraction of Expressions

Simplify by adding or subtracting as directed.

1. $5x + 3x + 7x$
2. $7y - 3y + 5y$
3. $2x^2 - 3x^2 + 5x^2$
4. $3ab + 2ab - 7ab$
5. $2x - 5 + 3x + 6$
6. $6t + 2 - 3t - 7$
7. Add:
$$\begin{array}{r} 2x - 5y + 3 \\ 3x + 6y - 4 \\ -4x - 2y + 2 \\ \hline \end{array}$$
8. Add:
$$\begin{array}{r} 2x^2 - 3xy + y^2 \\ -3x^2 - 2xy \\ x^2 + xy - 2y^2 \\ \hline \end{array}$$
9. Subtract:
$$\begin{array}{r} 5r - 6s \\ 3r + 2s \\ \hline \end{array}$$
10. Subtract:
$$\begin{array}{l} 3x^2 - 4y^2 - 6xy \\ 2x^2 + y^2 \\ \hline \end{array}$$
11. Arrange in *descending powers* of a and add:
$2a^2 - 3a + a$; $-4a + 3a^2 - 6$; $7 - 5a^2 + 2a$

4. A **monomial or term** consists of numbers connected *only* by signs of multiplication or division; as, $2x^2$, or $3ab$.

5. If two or more numbers are multiplied together, each of them or the product of any number of them is a **factor** of the product. A factor of a product is an exact divisor of the product.

6. Any factor of a product is called the **coefficient** of the product of the remaining factors.

Thus: In $2ab$, *2 is the coefficient of* ab, $2a$ *of* b, *and* $2b$ *of* a.

In a term like $5x^2y$, the factor 5 is called the **numerical coefficient** of x^2y. A term like x has the numerical coefficient 1. A term like $-3a$ has the numerical coefficient -3, since $(-3) \times a = -3a$.

7. When one number, the **base**, is used as a factor two or more times, the result is a **power** of the base. An **exponent**, written above and at the right of the base, indicates the number of times the base is used as a factor.

Thus: $3^4 = 3 \cdot 3 \cdot 3 \cdot 3$, or 81; $x^3 = x \cdot x \cdot x$; $y^5 = y \cdot y \cdot y \cdot y \cdot y$.

8. A **polynomial** consists of two or more terms.

A **binomial** is a polynomial having two terms.

A **trinomial** is a polynomial having three terms.

9. Like terms are terms which have the same literal factors.

Thus: $2x^2y$ and $-5x^2y$ are like terms.

Unlike terms are terms which do not have the same literal factors. Thus: $2x^2y$ and $2xy^2$ are unlike terms.

Unlike terms may be *like with respect* to one or more numbers; thus: axy and $2bxy$ are like with respect to x and y.

10. The laws of addition.

(*a*) **The associative law of addition.** The sum of three or more numbers is the same in whatever manner the numbers are grouped.

Thus: $a + b + c = (a + b) + c = a + (b + c)$.

(*b*) **The commutative law of addition.** The sum of two or more numbers is the same in whatever manner the numbers are arranged.

Thus: $a + b + c = a + c + b = c + b + a$.

11. Addition and subtraction of expressions.

Rule 1. To add two or more like terms:

Multiply their common factor by the sum of its coefficients.

Thus: $2a(x - y) + 3b(x - y) = (2a + 3b)(x - y)$.

Rule 2. To add polynomials:

1. *Write the polynomials with like terms in vertical columns.*
2. *Add the columns of like terms, and connect the results by their signs.*

Rule 3. To subtract one term from a like term or one polynominal from another:

1. *Write like terms in vertical columns.*
2. *Imagine the signs of the terms of the subtrahend changed, and add the resulting terms to those of the minuend.*

EXERCISE 2. REMEDIAL PRACTICE

1. Arrange in descending powers of x and add: $4x^2 - 5y^2 + 3xy$, $7xy - 2x^2 + 6y^2$, and $9y^2 - 3x^2 - 8xy$.

2. What is the numerical coefficient of $5n(x - y)$?

3. (*a*) Are $3c(x + y)$ and $4d(x + y)$ like in any respect? (*b*) What are the coefficients of the common factor?

4. Add $3(c - d) - 9(a + d)$, and $4(c - d) + 7(a + b)$.

5. Subtract $3s^3 - 10 + 5s - 2s^2$ from $8s^3 - s - 4s^2 + 11$.

6. How much more than $x^2 - 2x + 3$ is $3x^2 + 11x - 7$?

7. From $3a^2 - 4ab - 5b^2$ subtract the sum of $a^2 + ab - b^2$ and $2a^2 + 4ab - 5b^2$.

8. What must be added to $m^2 - 3m - 5$ to give $7m^2 + 4m - 3$?

9. Subtract $9(m + n) - 3(m - n)$ from $12(m - n) + (m + n)$.

10. From $3a^{2x} - 2a^xy^b + y^{2b}$ subtract $a^{2x} + a^xy^b - 3y^{2b}$.

11. Add $3x^{2n} - 2x^n + 1$ and $5x^{2n} + 3x^n - 4$.

12. Add $ax + by + cz$ and $Ax - sy + tz$.

DIAGNOSTIC TEST 3

Parentheses Used in Addition and Subtraction

Remove the parentheses and combine terms.

1. $x + (x + 5)$
2. $y + (y - 4)$
3. $z - (z + 6)$
4. $w - (w - 7)$
5. $2x - (-x + 5)$
6. $x - (-y - z)$
7. $3a + (-a - b)$
8. $4m - (-2m + n)$
9. $a + (b + (c - d))$
10. $a + (b - (c - d))$
11. $r - (s + (t - x))$
12. $r - (s - (t + x))$
13. $2x - (3x - (4x - 1))$
14. $5c + (4 - (2c - 3))$
15. $2m - (-m - (m - 1))$
16. $7c + (-2c + (-c + 3))$

12. Parentheses used in addition and subtraction.

The expression $x + (y - z)$ means that $y - z$ must be added to x. Addition of a number is accomplished by writing it with its own sign;

so $$x + (y - z) = x + y - z.$$

The expression $x - (y - z)$ means that $y - z$ must be subtracted from x. Subtraction of a number is accomplished by writing it with its sign changed;

so $$x - (y - z) = x - y + z.$$

If your teacher wishes, use the following customary rules.

Rule 1. When removing parentheses preceded by a plus sign, *do not change the signs of the terms within the parentheses;* **when removing parentheses preceded by a minus sign,** *change the signs of the terms within the parentheses.*

Rule 2. When placing terms within parentheses preceded by a plus sign, *do not change the signs of the terms;* **when placing terms within parentheses preceded by a minus sign,** *change the signs of the terms.*

Note. Remedial Practice and other examples appear on p. 237.

DIAGNOSTIC TEST 4

Multiplication of Algebraic Expressions

1. What is the meaning of x^2? of y^3? of $(mn)^4$?
2. How may $x \cdot x \cdot x \cdot x \cdot x$ be written more briefly?
3. Find: (*a*) $y^2 \cdot y^5$; (*b*) $x^4 \cdot x^5$; (*c*) $m \cdot m^4$.
4. Find: (*a*) $2x \cdot 3y$; (*b*) $4r \cdot 3r^2$; (*c*) $2xy \cdot 3yz$.
5. Find: (*a*) $(-2c) \cdot (+5a^2)$; (*b*) $(-4m^2) \cdot (-3mn)$.
6. Multiply $2a + 3b$ by ab.
7. Find the product of $2x - 3y$ and $4x$.
8. Find: (*a*) $5r(2s - 3t)$; (*b*) $-3c(2c - 5)$.
9. Find: (*a*) $x^2(x^2 - xy + y^2)$; (*b*) $(-2x)(3x^2 + x - 1)$.
10. Multiply $3x + 2y$ by $2x - 3y$.

13. The fundamental laws for multiplication.

(*a*) **The associative law of multiplication.** The product of three or more numbers is the same in whatever manner the numbers are grouped.

Thus: $a \cdot b \cdot c = (a \cdot b) \cdot c = a \cdot (b \cdot c).$

(*b*) **The commutative law of multiplication.** The product of two or more numbers is the same in whatever manner the numbers are arranged.

Thus: $a \cdot b \cdot c = b \cdot a \cdot c = c \cdot b \cdot a.$

(*c*) **The distributive law of multiplication.** If the sum or the difference of two or more numbers is multiplied by a number, the product may be found by multiplying each of the numbers by the multiplier and connecting the results by their signs.

Thus: $a(b + c - d) = ab + ac - ad.$

14. The law of exponents in multiplication.

Since $x^3 = x \cdot x \cdot x$ and $x^4 = x \cdot x \cdot x \cdot x,$

then $x^3 \cdot x^4 = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x,$ or $x^7.$

The exponent of any number in a product equals the sum of its exponents in the factors of the product.

15. Rule. To find the product of two monomials:

1. *Find the product of their numerical coefficients.*
2. *Multiply this result by the product of the literal factors, using the law of exponents for multiplication.*

Example. $(3\,x^2y)(-4\,yz^3) = -12\,x^2y^2z^3$

16. Rule. To find the product of a polynomial and a monomial:

1. *Multiply each term of the polynomial by the monomial.*
2. *Unite the results with their signs.*

This rule is an application of fundamental law (*c*) of § 13.

Example. $(-3\,a)(x + y - z) = -3\,ax - 3\,ay + 3\,az.$

17. Rule. To find the product of a polynomial and a polynomial:

1. *Multiply the multiplicand by each term of the multiplier.*
2. *Add the partial products.*

It is desirable to arrange both the multiplier and the multiplicand in *descending* or *ascending powers* of one number.

Example. Multiply $2x - 3y$ and $x^2 - 5y^2 - 3xy$.

Solution.

$$\begin{array}{rrrrl} x^2 & -\ 3xy & -\ 5y^2 & & \\ & 2x & -\ 3y & & \\ \hline 2x^3 & -\ 6x^2y & -\ 10xy^2 & & \text{This is } 2x(x^2 - 3xy - 5y^2) \\ & -\ 3x^2y & +\ 9xy^2 & +\ 15y^3 & \text{This is } -3y(x^2 - 3xy - 5y^2) \\ \hline 2x^3 & -\ 9x^2y & -\ xy^2 & +\ 15y^3 & \end{array}$$

EXERCISE 3. REMEDIAL PRACTICE

1. Find: *a*) $x^3 \cdot x^2$; *b*) $y^4 \cdot y^3$; *c*) $z^c \cdot z^2$; *d*) $10^2 \cdot 10^3$
2. Find: *a*) $(-2x)(+3x^2)$; *b*) $(xy)(-8yz)$; *c*) $(-a)(-a)$

Multiply:

3. $-11c^2$ by $+5c^3d$
4. $-3x^2y^3$ by $-12xy^2z$
5. $3a^2$ by a
6. $-m^2n$ by $5mn$
7. $-xy$ by $-x$
8. $3a^2 - 2a$ by $5a$
9. $x^2 - xy$ by $2xy$
10. $x^2 - x - 3$ by -1
11. $m^2 + 2mn + n^2$ by $m + n$
12. $c^2 - 2cd + d^2$ by $c + d$
13. $a^2 - ab - 2b^2$ by $2a - b$
14. $x^2 - 3x + 4$ by $2x - 3$
15. $x^3 + x - 1$ by $x - 2$
16. $y^3 - 2y^2 + 3$ by $y - 4$
17. $x^2 - xy^2$ by $x + y$
18. $4a^4 + 2a^2 + 1$ by $2a^2 - 1$

Multiply and then collect like terms:

19. $8\left(\frac{1}{2}x + \frac{1}{4}x\right)$
20. $10\left(\frac{y}{5} - \frac{y}{2} + \frac{3y}{2}\right)$
21. $12\left(\frac{a}{3} - \frac{a}{4} + \frac{a}{6}\right)$
22. $24\left(\frac{x}{6} - \frac{2x}{3} + \frac{x}{8}\right)$
23. $18\left(\frac{y}{9} - \frac{y}{3} + \frac{y}{2}\right)$
24. $15\left(\frac{1}{2}a - \frac{1}{3}a - \frac{1}{5}a\right)$

18. Parentheses used in multiplication.

Since $5x - 2(y - 3) = 5x + [-2(y - 3)]$

therefore

$$5x - 2(y - 3) = 5x + (-2y + 6), \text{ or } 5x - 2y + 6.$$

We can get this result immediately if we multiply y by -2, and -3 by -2.

Example 1. $2x - 3(x - 2) = 2x - 3x + 6$, or $6 - x$.

Here x is multiplied by -3; also -2 is multiplied by -3.

Example 2. Multiply $(2x - 3)$ and $x^2 - 3x - 5$.

Solution. 1. $(2x - 3)(x^2 - 3x - 5)$

2. $= 2x(x^2 - 3x - 5) - 3(x^2 - 3x - 5)$

3. $= 2x^3 - 6x^2 - 10x - 3x^2 + 9x + 15$

4. $= 2x^3 - 9x^2 - x + 15.$

NOTE. In Step 3, $-3x^2$ is $(-3)x^2$; $+9x$ is $(-3)(-3x)$; etc.

EXERCISE 4, *a*

Simplify as in Examples 1 and 2 above.

1. $6a - 2(a + 1)$
2. $7c - 3(2 - c)$
3. $5m - 4(m + 3)$
4. $8x - 5(2 + x)$
5. $4(2c - 1) - 3(c + 1)$
6. $2(5 - x) - 2(3x - 5)$
7. $6(3 + 2t) - (5t + 11)$
8. $8(2 - s) - 9(1 - 2s)$
9. $-3(x - 2) + 4(x - 1)$
10. $\frac{1}{2}(4r - 6) - (9 - 3r)$
11. $(x + 3)(3x^2 - 2x + 5)$
12. $(m - 2n)(2m^2 + mn - 3n^2)$
13. $(3r - s)(r^2 - 4rs - s^2)$
14. $(2x - 3y)(x^2 - 7xy - 2y^2)$
15. $(a^2 - a - 3)(a - 2)$
16. $(2a^2 - a - 3)(a - 2)$
17. $(3c^2 + 7c - 9)(5c - 1)$
18. $(x^2 + 4xy + 16y^2)(x - 4y)$
19. $(m^2 - mx - 6x^2)(5m - 2x)$
20. $(2x - 1)(x^2 - 4x - 5)$

EXERCISE 4, *b*

1. $x^n(x + 1)$
2. $-y^2(y^n - 2)$
3. $x^a(x^{2a} - x^a + 1)$
4. $-3x^c(x^{2c} - 2x^c - 1)$
5. $5z^m(z^{2m} - z^m - 3)$
6. $-4a^x(-1 + a^x - a^{2x})$
7. $(x^n - 5)(x^n - 4)$
8. $(a^x + b^y)(a^x - b^y)$

DIAGNOSTIC TEST 5

Division of Algebraic Expressions

1. Find: *a.* $y^5 \div y^2$; *b.* $x^6 \div x^4$; *c.* $a^7 \div a^2$
2. Find: *a.* $4x^2 \div 2x$; *b.* $12x^6 \div 3x^2$; *c.* $10xy^4 \div 2xy$
3. Find: *a.* $-6m^4 \div 2m$; *b.* $(+8ab^2) \div (-4ab)$; *c.* $(-14c^3) \div (-2c)$
4. Divide: $6x + 8y$ by 2
5. Divide: $12x^2 - 6x^3 + 4x^4$ by $2x^2$
6. Divide: $-4a^2 + 8a$ by $-2a$
7. Find: *a.* $\dfrac{6x - 4y}{2}$ *b.* $\dfrac{-x^2 + x - 1}{-1}$ *c.* $\dfrac{x^3 - x^2 + x}{x}$
8. Divide: $x^2 - 7x + 12$ by $x - 4$
9. Divide: $6x^2 + 11xy - 10y^2$ by $2x + 5y$
10. Divide: $x^3 - 4x^2 + 7x - 6$ by $x - 2$

√ **19.** In **division**, the number divided is the **dividend**; the number by which it is divided is the **divisor**; the result is the **quotient**. If the division is not exact, there is a remainder. Always *dividend* = *divisor* × *quotient* + *remainder*.

20. Division by zero is impossible.

For if $n \div 0 = m$, then $m \cdot 0 = n$. But $m \cdot 0 = 0$, and not n.

21. Division of any number by itself equals 1.

Thus $n \div n = 1$, for $n \cdot 1 = n$.

22. The law of exponents in division.

Since $$\frac{x^5}{x^2} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x} = 1 \cdot 1 \cdot x \cdot x \cdot x,$$

then $$x^5 \div x^2 = x^3.$$

That is, *the exponent of any number in a quotient equals its exponent in the dividend minus its exponent in the divisor.*

Thus: $x^7 \div x^3 = x^{7-3}$, or x^4.

Again: $yz^2 \div yz = \dfrac{yz^2}{yz} = \left(\dfrac{y}{y}\right)\left(\dfrac{z^2}{z}\right) = 1 \cdot z$, or z.

23. Rule. To divide a monomial by a monomial:

1. *Find the quotient of their numerical coefficients.*

2. *Multiply the result by the product of their literal factors, using the law of exponents for division.*

Thus: $$(-12x^2y^3) \div (2xy^3) = -6x$$

24. Rule. To divide a polynomial by a monomial:

1. *Divide each term of the polynomial by the monomial.*

2. *Unite the results with their signs.*

Thus: $$(6a^3 - 4a^2 - 2a) \div 2a = 3a^2 - 2a - 1$$

25. Rule. To divide a polynomial by a polynomial:

1. *Arrange the dividend and the divisor in either ascending or descending powers of some common letter.*

2. *Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.*

3. *Multiply the whole divisor by the first term of the quotient; subtract the product from the dividend.*

4. *Using the remainder as a new dividend, repeat Steps 1–3.*

Example. Divide $2a^4 + 8a - a^3 + 15$ by $2a^2 - 3a + 5$.

Solution.

$$\begin{array}{r|l} & a^2 + a - 1 \\ \hline 2a^2 - 3a + 5 & 2a^4 - a^3 \qquad + 8a + 15 \\ & 2a^4 - 3a^3 + 5a^2 \\ \hline & \qquad 2a^3 - 5a^2 + 8a + 15 \\ & \qquad 2a^3 - 3a^2 + 5a \\ \hline & \qquad\quad -2a^2 + 3a + 15 \\ & \qquad\quad -2a^2 + 3a - 5 \\ \hline & \qquad\qquad\qquad +20 \end{array}$$

Check. "*Dividend = divisor × quotient + remainder.*"

$$\begin{array}{l} 2a^2 - 3a + 5 \\ \underline{a^2 + a - 1} \\ 2a^4 - 3a^3 + 5a^2 \\ \quad + 2a^3 - 3a^2 + 5a \\ \qquad\quad \underline{- 2a^2 + 3a - 5} \\ 2a^4 - a^3 \qquad + 8a - 5 \end{array}$$

$$\begin{array}{r} 2a^4 - a^3 + 8a - 5 \\ \underline{+20} \\ 2a^4 - a^3 + 8a + 15 \end{array}$$

Since this is the dividend, the division was performed correctly.

EXERCISE 5. REMEDIAL PRACTICE

Divide and check your solutions:

1. $-10\,xy^2$ by $+5\,xy$
2. $-24\,a^4b^2c$ by $-8\,a^4b^2$
3. $28\,ab^3c^5$ by $-7\,b^2c^5$
4. $2\,a^3 - a^2 + a$ by a
5. $6\,x^4 - 3\,x^3 + 9\,x^2$ by $-3\,x^2$
6. $-44\,a^2b + 55\,a^3b$ by $11\,a^2b$
7. $-3\,x^4 + 6\,x^2 - x^3$ by $-x^2$
8. $21\,x^3y^2 - 42\,xy^3$ by $-7\,xy^2$
9. $18\,ax^4 - 50\,ay^4$ by $2\,a$
10. $mx^2 - 6\,mx + 9\,m$ by m
11. $15\,x^2 - 11\,x - 14$ by $3\,x + 2$
12. $32\,x^2 - 15\,y^2 + 28\,xy$ by $4\,x + 5\,y$
13. $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$
14. $4\,x^2 - y^2 + 2\,yz - z^2$ by $2\,x - y + z$
15. $x^3 - 8\,y^3$ by $x - 2\,y$
16. $c^4 - d^4$ by $c + d$
17. $c^5 + d^5$ by $c + d$
18. $1 - 81\,m^4$ by $1 + 3\,m$

EXERCISE 6

Divide:

1. $4\,x^{2n}$ by $-x^n$
2. $-14\,r^{4x}$ by $7\,r^x$
3. $24\,a^{2m}b^n$ by $-2\,a^mb^n$
4. $-12\,x^{3a}y^{2b}$ by $-4\,x^ay^b$
5. $24\,a^{2m+3}$ by $-4\,a^{2m}$
6. $-14\,x^{2n}y^{3m}$ by $-2\,x^2y^3$
7. $a^{3x} + 2\,a^{2x} - a^x$ by a^x
8. $c^6 - c^4 + c^2$ by c^x
9. $8\,y^t - 12y^{2t} + 16\,y^{3t}$ by $4\,y^t$
10. $15\,x^m - 10\,x^n - 5\,x^r$ by $-5\,x^n$

11. *a.* Find the sum, the product, and the quotient of $-2\,x^cy^d$ and $5\,x^cy^d$.

 b. Also of $-3(a - b)$ and $+4(a - b)$.

12. Simplify $x^n(x^n - 2) + 2(x^n - 1)$
13. Simplify $3(.2\,a + .3\,b) - 4(.1\,a - .5\,b)$
14. Find $(3\,x - 5\,y)^2 - 5\,y(x - 5\,y)$
15. Find $(x + a - b)(x - a + b)$
16. Divide $x^{2p} + 3\,x^p - 54$ by $x^p - 6$

26. The order of performing the fundamental operations.

In $2 + 3 \times 4$, if 2 and 3 are added, securing 5, and the result is multiplied by 4, the final result is 20. But, mathematicians have agreed that in such an expression, the multiplication should be done first and afterward the addition;

so $$2 + 3 \times 4 = 2 + 12, \text{ or } 14.$$

If the other interpretation were desired, then the expression should have been written thus: $(2 + 3) \times 4$.

By means of parentheses one can always clearly indicate which operations are to be done first. The following rule should be used when more than one of the fundamental operations is to be performed, in case parentheses are not used to indicate some other order of performing them.

Rule. 1. *First perform all operations within parentheses or under radical signs.*

2. *Next do all the multiplications.*

3. *Next do all the divisions.*

4. *Finally do all the additions and subtractions in the order in which they occur.*

Example 1. Find the value of $I \div rt$ when $I = \$100$, $r = .05$, and $t = 4$.

Solution.
$$\begin{aligned} I \div rt &= 100 \div .05 \times 4 \\ &= 100 \div .2, \text{ or } \$500 \end{aligned}$$

Most books advise doing the multiplications and the divisions in order as they occur. This example would give $(100 \div .05) \times 4$, or 2000×4, or \$8000. Since the formula used in Example 1 gives the principal which produces \$100 interest at 5% in 4 years, this second result is absurd, for $\$8000 \times .05 \times 4 = \$1600$.

Example 2. If $h = V \div \pi r^2$, find h when $V = 770$, $\pi = 3\frac{1}{7}$, and $r = 7$.

Solution. $h = 770 \div \frac{22}{7} \cdot 7 \cdot 7$; $\therefore h = 770 \div 154$, or $h = 5$.

II. SPECIAL PRODUCTS AND FACTORING

27. The following diagnostic test will help you discover what you remember about the simple cases in factoring taught in the first course in algebra. Make certain that you are skillful in these simple cases.

DIAGNOSTIC TEST 6

Do mentally and write *only* the results for:

1. $(x + 6)(x - 6)$

2. $(x + 5)(x + 7)$

3. $(y - 3)(y - 9)$

4. $(2x - 5)(2x + 5)$

5. $(m + 11)(m - 4)$

6. $(6 - z)(10 + z)$

7. $(2a + 1)(a + 2)$

8. $(4c - 3)(c + 2)$

9. $(x - 5y)(3x + 2y)$

10. $(\frac{2}{3}x - 1)(\frac{2}{3}x + 1)$

11. $(3a + 2)(3a + 2)$

12. $(5x - 3y)(2x + 3y)$

Factor the following expressions:

13. $x^2 - 25$

14. $x^2 + 6x + 9$

15. $y^2 - 8y + 16$

16. $49 - m^4$

17. $c^2 + 5c + 6$

18. $m^2 - 7m + 12$

19. $x^2 - 4x - 21$

20. $z^2 + 6z - 16$

21. $w^2 + 4wt - 45t^2$

22. $2a^2 + 5a + 2$

23. $3b^2 - 8b + 4$

24. $6c^2 - 11c + 3$

25. $2x^2 + x - 6$

26. $3y^2 + 4y - 4$

27. $10w^2 - 3w - 1$

28. $7x^2 - 11x - 6$

29. $12t^2 + 16t - 3$

30. $15r^2 - 4r - 4$

NOTE. Your score on this test will indicate which parts of pages 16 to 20 you must study with greatest care.

28. The product of two binomials of the form $ax + b$.

Observe two ways of finding $(3x + 2y)(2x - 5y)$.

Solution A.

$$\begin{array}{r} 3x + 2y \\ 2x - 5y \\ \hline 6x^2 + 4xy \qquad\qquad \\ -15xy - 10y^2 \\ \hline 6x^2 - 11xy - 10y^2 \end{array}$$

Solution B.

$$(3x + 2y)(2x - 5y)$$

$$+4xy$$

$$-15xy$$

$$= 6x^2 - 11xy - 10y^2$$

Such products are to be found always as in Solution B.

Observe: 1. $6x^2$ is the product of $3x$, and $2x$, the first terms.

2. $-11xy$ is the sum of $(+4xy)$ and $(-15xy)$.

Notice that $+4xy$ is the product of the two terms which are *close together;* that $-15xy$ is the product of the two which are *far apart.* (See the arrows in Solution B.)

These products are usually called the *cross products* because, in Solution A, the arrow connecting $(2x)$ and $(+2y)$ crosses the arrow connecting $3x$ and $(-5y)$.

3. $-10y^2$ is the product of the *second terms.*

Rule. *The product of two binomials of the form $ax + b$ is the product of their first terms, plus the algebraic sum of their cross products, plus the product of their second terms.*

Example 1. $(2c - 3d)(3c + 5d) = 6c^2 + cd - 15d^2$

[$(-3d)(3c) = -9cd$; $(2c)(+5d) = +10cd$;
$(-9cd) + (+10cd) = +cd$.]

Example 2. $(3x - 5)(3x + 5) = 9x^2 - 25$

[The middle term drops out since $(-15x) + (+15x) = 0$.]

Example 3. $(2c + 5d)^2 = (2c + 5d)(2c + 5d)$
$= 4c^2 + 20cd + 25d^2$

[The middle term is twice $(2c)(+5d)$ since $(10cd) + (10cd) = 20cd$.]

Examples 2 and 3 can be solved as in § 29 if your teacher wishes. § 29 can be studied before doing Remedial Practice 7.

EXERCISE 7. REMEDIAL PRACTICE

Find the following products mentally:

1. $(3x + 2)(x - 4)$
2. $(6a + b)(3a - 2b)$
3. $(9c - 5d)(2c + d)$
4. $(5a - 4b)(5a + 4b)$
5. $(n^3 - 3m)(n^3 + 3m)$
6. $(r^2 - 6s)(r^2 + 2s)$
7. $(x - 2y^2)(x + 2y^2)$
8. $(2x + 3y)(2x - 4y)$
9. $(5a^2 + b^2)(3a^2 + 2b^2)$
10. $(3c + d)^2$
11. $(4a - b)^2$
12. $(2x - 3y)^2$
13. $(m - 11n)(m + 8n)$
14. $(3x^2 - y^3)(x^2 + 4y^3)$
15. $(b^2 - 3ac)(b^2 + 5ac)$
16. $(x - \frac{3}{5})(x + \frac{3}{5})$
17. $(7a^2 - b)(6a^2 + b)$
18. $(\frac{1}{2}a - 3b)(\frac{1}{2}a + 3b)$
19. $(x^2y + 7z)(x^2y - 12z)$
20. $(3c - 2d^2)^2$
21. $(s - 11t)(s + 3t)$
22. $(m^2 - 4n)(m^2 + 4n)$
23. $(2ab - 5)^2$
24. $(x - \frac{1}{2})(x + \frac{1}{4})$
25. $(3x + 2)(x - 11)$
26. $(y + 10z)(y - 4z)$
27. $(\frac{1}{3}s - 4t)(\frac{1}{3}s + 4t)$
28. $(1 - 5A^2)(2 + 9A^2)$
29. $(2c - 5w^2)(3c + 5w^2)$
30. $(\frac{3}{5}m - \frac{2}{3})(\frac{3}{5}m + \frac{2}{3})$
31. $(4p - 7r)(2p + 3r)$
32. $(12s^3 - 1)(12s^3 + 1)$
33. $(z^2 - 5m)(z^2 + 11m)$
34. $(5x^3 - 2y^2)(5x^3 + y^2)$
35. $(7y + \frac{1}{3})(7y - \frac{1}{3})$
36. $(4a - 3b)^2$
37. $(4m - 7n)(2m + 5n)$
38. $(2 - 5yz)(4 + 9yz)$
39. $(x^2y + 3a)(x^2y - 5a)$
40. $(x - 8c)(x + 10c)$
41. $(3x + 5y)(2x - 7y)$
42. $(a - 9t^2)(a + 8t^2)$
43. $(3c - 2d^2)(2c + 5d^2)$
44. $(x - \frac{2}{3})(x + \frac{2}{3})$
45. $(y - \frac{2}{3})^2$
46. $(6a + 5b)(3a - 4b)$
47. $(z - \frac{3}{5})^2$
48. $(12x - 9y)(12x + 9y)$
49. $(w - \frac{3}{4})^2$
50. $(8m - 5n)(3m + 2n)$
51. $(2a - \frac{1}{2})^2$
52. $(11c^2 - 8d)(11c^2 + 8d)$
53. $(3x - \frac{1}{5})^2$
54. $(9s + 4t)(3s - 2t)$
55. $(\frac{3}{5}x - \frac{2}{3}y)(\frac{3}{5}x + \frac{2}{3}y)$
56. $(m - \frac{3}{7})^2$

29. The following two special cases of the rule taught in § 28 save time, and are easily remembered.

Rule 1. The product of the sum and the difference of the same two numbers *equals the difference of their squares.*

$$(x + y)(x - y) = x^2 - y^2$$

Rule 2. The square of a binomial *equals the square of its first term, plus twice the product of its two terms, plus the square of its second term.*

$$(x + y)^2 = x^2 + 2\,xy + y^2$$

Example 1.

$$(3\,c - 4\,d)^2 = (3\,c)^2 + 2\,(3\,c)(-\,4\,d) + (-\,4\,d)^2$$
$$= 9\,c^2 - 24\,cd + 16\,d^2$$

Example 2.

$$[(x - y) - z][(x - y) + z] = (x - y)^2 - z^2$$
$$= x^2 - 2\,xy + y^2 - z^2$$

EXERCISE 8. REMEDIAL PRACTICE

Find the following products as above. Consider examples 16 to 24 optional.

1. $(2\,a + 5)^2$
2. $(3\,a - 2)^2$
3. $(1 + 4\,c)^2$
4. $(x - 3\,y)^2$
5. $(\frac{1}{2}\,a + b)^2$
6. $(c - \frac{1}{3}\,d)^2$
7. $(x - 6\,t)(x + 6\,t)$
8. $(3\,a - 5\,b)(3\,a + 5\,b)$
9. $(2\,c + 1)(2\,c - 1)$
10. $(x^2 - y^2)^2$
11. $(2\,a^2 + 3\,b)^2$
12. $(x^2 - y^3)(x^2 + y^3)$
13. $(\frac{1}{2}\,m^2 + n)(\frac{1}{2}\,m^2 - n)$
14. $(c^3 - \frac{1}{3})(c^3 + \frac{1}{3})$
15. $(3\,cd + 2)^2$
16. $[(x + y) + z][(x + y) - z]$
17. $[(r + s) - 2][(r + s) + 2]$
18. $[(a + 3) + b][(a + 3) - b]$
19. $[(m - 5) - n][(m - 5) + n]$
20. $[(a + b) + 1]^2$
21. $[2 + (y + z)]^2$
22. $[(c - d) + 3]^2$
23. $[(x - y) + z]^2$
24. $[(r - s) - t]^2$

30. Factoring trinomials of the form $ax^2 + bx + c$.

Example. Factor $15x^2 + 17x - 4$.

Solution. 1. We seek two binomials whose product is $15x^2 + 17x - 4$.

2. Assume the first terms are $5x$ and $3x$; thus: $(5x \quad)(3x \quad)$.

3. Assume the second terms are 2 and 2, since $2 \times 2 = 4$. Arrange the signs so that the larger cross product is plus; that is, try $(5x - 2)(3x + 2)$. Since the middle term of this product is $+4x$, these are not the correct factors.

4. Assume the second terms are 4 and 1, since $4 \times 1 = 4$. Arrange the signs as before, and try $(5x + 4)(3x - 1)$. This is not correct for the middle term now is $+7x$.

5. Rearrange and try $(5x - 1)(3x + 4)$. This is correct, for the middle term now is $+17x$.

6. Hence $\quad 15x^2 + 17x - 4 = (5x - 1)(3x + 4)$.

EXERCISE 9. REMEDIAL PRACTICE

Factor the following trinomials:

1. $x^2 + 10x + 24$
2. $y^2 + 11y + 28$
3. $z^2 - 11z + 30$
4. $w^2 - 9w + 18$
5. $r^2 + 4r - 21$
6. $s^2 - 2s - 15$
7. $t^2 + t - 20$
8. $m^2 - 5m - 24$
9. $3x^2 + 5x + 2$
10. $2w^2 - 11w + 5$
11. $5m^2 + 7m + 2$
12. $6m^2 - 7m + 2$
13. $2w^2 - 9w + 4$
14. $5c^2 - 23c + 12$
15. $8a^2 - 22a + 15$
16. $2x^2 - 3x - 5$
17. $3m^2 + 4m - 7$
18. $5x^2 - 2x - 7$
19. $7c^2 - 4c - 11$
20. $6t^2 + 7t - 5$
21. $12y^2 - 5y - 3$
22. $2z^2 - z - 15$
23. $9t^2 + 6t - 8$
24. $15x^2 + 4x - 3$
25. $5a^2 + 16ab + 3b^2$
26. $4x^2 - 24xy + 35y^2$
27. $3t^2 - 22ts + 7s^2$
28. $4r^2s^2 + 4rs - 15$
29. $12x^2 - 17x + 6$
30. $16x^2 - 24xy + 9y^2$

NOTE. Additional examples appear on p. 238.

31. Two important special cases of factoring.

Rule 1. The difference of two squares *equals the product of the sum and the difference of their square roots.*

$$(x + y)(x - y) = x^2 - y^2$$

Rule 1. A perfect square trinomial *has two terms which are perfect squares, and a third which is the product of their square roots.*

2. *It has two identical factors. Each consists of the square roots of the perfect square terms connected by the sign of the third term.*

Thus: $4x^2 - 12xy + 9y^2 = (2x)^2 - 2 \cdot (2x)(3y) + (3y)^2$
$= (2x - 3y)(2x - 3y)$

EXERCISE 10. REMEDIAL PRACTICE

1. $y^2 - 64$
2. $x^2 - 10x + 25$
3. $z^2 + 6az + 9a^2$
4. $25m^4 - 1$
5. $4c + 4cd + d^2$
6. $16 - 8x + x^2$
7. $w^2 - \frac{1}{4}y^2$
8. $25a^2 + 10a + 1$
9. $49b^2 - 14b + 1$
10. $9c^2 - \frac{4}{25}$
11. $c^4 - 12c^2 + 36$
12. $16x^2 - 25y^2$
13. $w^2 - 16w + 64$
14. $36 - 12s + s^2$
15. $c^2 + d^2 - 2cd$
16. $y^2 + 16y + 64$
17. $9x^4 - 1$
18. $\frac{4}{9}a^2 - b^2$
19. $1 - 6b + 9b^2$
20. $\frac{1}{16} - m^4$
21. $36x^2 - 121$
22. $100a^2 - 20ab + b^2$
23. $9s^2 - 12st + 4t^2$
24. $\frac{1}{9}m^2 - \frac{1}{16}$
25. $1 - \frac{1}{25}t^2$
26. $z^2 + \frac{2}{5}z + \frac{1}{25}$
27. $x^2 - \frac{4}{3}x + \frac{4}{9}$
28. $x^2 - \frac{2}{3}x + \frac{1}{9}$
29. $y^2 + y + \frac{1}{4}$
30. $z^2 - .25$
31. $w^2 - .36$
32. $.49x^2 - 1$
33. $y^2 - .09z^2$
34. $w^2 - .12w + .36$
35. $x^2 - .14x + .49$
36. $.64 + .16x + x^2$
37. $.04x^2 - 1$
38. $y^2 - .09z^2$
39. $.04x^2 - .04x + 1$
40. $y^2 - .06y + .09$

SOME NEW TOPICS IN FACTORING, PAGES 21 TO 28

32. Polynomials factored by grouping (*Optional*).

Example 1. Just as $ax + bx = (a + b)x$,

so $$a(x + y) + b(x + y) = (a + b)(x + y).$$

Example 2. Factor $6x^3 - 15x^2 - 8x + 20$.

Solution. 1. $6x^3 - 15x^2 - 8x + 20$
$= (6x^3 - 8x) - (15x^2 - 20)$

2. $= 2x(3x^2 - 4) - 5(3x^2 - 4)$

3. $= (2x - 5)(3x^2 - 4)$

EXERCISE 11

Factor:

1. $2m(x + y) - 3(x + y)$
2. $5c(r - s) + 2d(r - s)$
3. $4n(2x - y) - b(2x - y)$
4. $9(p - q) - x(p - q)$
5. $a(b - c) - d(b - c)$
6. $ay + az + my + mz$
7. $mr - ms + nr - ns$
8. $ax - ay - bx + by$
9. $x^3 + x^2 + x + 1$
10. $y^3 + y^2 - 3y - 3$
11. $2a^3 - 5a - 2a^2 + 5$
12. $z^3 - 4z + z^2 - 4$
13. $m^3 + 3m^2 - 2m - 6$
14. $s^3 + 15 - 5s - 3s^2$
15. $x^2y - 2xy + 4x - 8$
16. $2cd - c^2d - 6 + 3c$
17. $ac + ad - 2bd - 2bc$
18. $ac - 3bc - ad + 3bd$
19. $mr - 3nr - 3ns + ms$
20. $2ax - 4ay - bx + 2by$
21. $5mr - 3ns - 3ms + 5nr$
22. $2 - 3a^3 - 2a^2 + 3a$
23. $3x^3 - 6x^2 + x - 2$
24. $2mx - 3nx - 2m + 3n$
25. $a^2x - a^2by + cx - bcy$
26. $a^2b - acd + bcd - ab^2$
27. $5a^3 - 55a - 2a^2 + 22$
28. $6 + 3x^2 - 8x - 4x^3$
29. $2ax + 3bx - 2ay - 3by$
30. $4x^2 - 2y^3 + xy^2 - 8yx$
31. $2cx + 4dx - 3cy - 6dy$
32. $x^3 + x^2y - xy^2 - y^3$
33. $m^3 - m^2n - mn^2 + n^3$
34. $ax - 2ay + 6by - 3bx$
35. $2x - x^3 + 3x^2 - 6$
36. $3y^3 + y^2 - 2 - 6y$
37. $2y^3 - zy + 2y^2z - z^2$
38. $2s^3 - 4s^2t + st^2 - 2t^3$

33. Polynomials reducible to the difference of two squares.

Example 1. Just as $x^2 - y^2 = (x + y)(x - y)$

so $$a^2 - (m - n)^2 = [a + (m - n)][a - (m - n)]$$
$$= [a + m - n][a - m + n]$$

Example 2. Factor $a^2 - b^2 + 2\,bc - c^2$

Solution. 1. $a^2 - b^2 + 2\,bc - c^2 = a^2 - (b^2 - 2\,bc + c^2)$

2. $= a^2 - (b - c)^2$

3. $\therefore\ a^2 - b^2 + 2\,bc - c^2 = [a + (b - c)][a - (b - c)]$

4. $= [a + b - c][a - b + c]$

EXERCISE 12

Factor:

1. $(a - b)^2 - c^2$
2. $x^2 - (z - y)^2$
3. $m^2 - (r + s)^2$
4. $(x + z)^2 - y^2$
5. $(2\,a - b)^2 - 9\,c^2$
6. $16\,s^2 + (t + w)^2$
7. $(x - 2\,y)^2 - 4\,z^2$
8. $25 - (r - 3\,s)^2$
9. $(r - 3)^2 - 1$
10. $9 - (2\,s + 3)^2$
11. $a^2 + 2\,ab + b^2 - c^2$
12. $m^2 - 2\,m + 1 - 4\,p^2$
13. $x^2 + 6\,x + 9 - y^2$
14. $t^2 - r^2 - 2\,rs - s^2$
15. $4\,c^2 - a^2 + 2\,ab + b^2$
16. $9\,x^2 - 6\,xy + y^2 - z^2$
17. $16\,a^2 + 8\,ab + b^2 - c^2$
18. $9\,r^2 - 4\,s^2 - 4\,s - 1$
19. $t^2 - 10\,t + 25 - 4\,x^2$
20. $16 - x^2 + 8\,x - 64$
21. $x^2 - 4\,xy + 4\,y^2 - z^2$
22. $b^2 - 6\,bc + 9\,c^2 - d^2$
23. $m^2 - 4\,r^2 - 8\,rs - 4\,s^2$
24. $9\,x^2 - z^2 - 2\,zw - w^2$
25. $9\,a^2 + 4\,bc - 4\,c^2 - b^2$
26. $x^2 + 6\,x - 16\,y^2 + 9$
27. $c^2 - b^2 - 4\,a^2 - 4\,ab$
28. $4 - 22\,xy - 121\,x^2 - y^2$
29. $16\,k^2 - 25\,n^2 + 1 - 8\,k$
30. $2\,AB + A^2 - T^2 + B^2$

Miscellaneous Factoring Examples

31. $(x^2 + 1)^2 - 4\,x^2$
32. $(r^2 - 3)^2 - (r + 1)^2$
33. $(x + 1)^2 - 5(x + 1) + 6$
34. $(r - 2)^2 + 7(r - 2) + 12$
35. $.25\,x^2 - 3\,x + 9$
36. $6\,x^2 + 5\,xy - 25\,y^2$
37. $\frac{4}{25}\,x^2 - \frac{9}{16}\,y^2$
38. $x^2 - \frac{6}{5}\,x + \frac{9}{25}$

34. Trinomials reducible to the difference of two squares.*

Type form: $x^4 + x^2y^2 + y^4$

Example. Factor $64\,a^4 - 64\,a^2m^2 + 25\,m^4$.

Solution. 1. *A perfect square* containing $64\,a^4$ and $25\,m^4$ is $64\,a^4 - 80\,a^2m^2 + 25\,m^4$. It can be obtained from $64\,a^4 - 64\,a^2m^2 + 25\,m^4$ by subtracting $16\,a^2m^2$. Then we must also add $16\,a^2m^2$ to avoid changing the value of $64\,a^4 - 64\,a^2m^2 + 25\,m^4$. That is,

2. $64\,a^4 - 64\,a^2m^2 + 25\,m^4 = (64\,a^4 - 80\,a^2m^2 + 25\,m^4) + 16\,a^2m^2$
$= (8\,a^2 - 5\,m^2)^2 + (4\,am)^2$

But this is not factorable, since it is the sum of two squares.

3. *Another perfect square* containing $64\,a^4$ and $25\,m^4$ is $64\,a^4 + 80\,a^2m^2 + 25\,m^4$. It may be obtained from $64\,a^4 - 64\,a^2m^2 + 25\,m^4$ by adding to it $144\,a^2m^2$. Then we must also subtract $144\,a^2m^2$.

That is, $64\,a^4 - 64\,a^2m^2 + 25\,m^4$
$= (64\,a^4 + 80\,a^2m^2 + 25\,m^4) - 144\,a^2m^2$

4. $= (8\,a^2 + 5\,m^2)^2 - (12\,am)^2$

5. $= \{(8\,a^2 + 5\,m^2) + 12\,am\}\{(8\,a^2 + 5\,m^2) - 12\,am\}$

6. $= \{8\,a^2 + 5\,m^2 + 12\,am\}\{8\,a^2 + 5\,m^2 - 12\,am\}$

EXERCISE 13

Factor:

1. $a^4 + a^2b^2 + b^4$
2. $m^4 + m^2 + 1$
3. $x^4 + 4\,x^2 + 16$
4. $y^4 + 9\,y^2 + 81$
5. $z^4 + 6\,z^2w^2 + 25\,w^4$
6. $r^4 + 3\,r^2s^2 + 36\,s^4$
7. $9\,m^4 + 2\,m^2t^2 + t^4$
8. $16\,a^4 - 12\,a^2 + 1$
9. $9\,x^4 - 10\,x^2 + 1$
10. $9\,a^4 + 2\,a^2b^2 + b^4$
11. $36\,t^4 - 16\,t^2 + 1$
12. $25\,x^4 - 19\,x^2y^2 + y^4$
13. $4\,a^4 + 8\,a^2 + 9$
14. $9\,x^4 + 20\,x^2 + 16$
15. $4\,a^4 + 11\,a^2b^2 + 25\,b^4$
16. $9\,c^4 - 34\,c^2d^2 + 25\,d^4$
17. $16\,a^4 - 28\,a^2b^2 + 9\,b^4$
18. $9\,r^4 - 43\,r^2t^2 + 49\,t^4$
19. $16\,x^4 + 4\,x^2y^2 + 25\,y^4$
20. $4\,m^4 + 19\,m^2n^2 + 49\,n^4$
21. $x^4 + 4$
22. $4\,a^4 + 1$
23. $c^4 + 4\,d^4$
24. $4\,x^4 + y^4$
25. $r^4 + 64\,t^4$
26. $x^4 + 324$

* This type is not required very generally by colleges.

35. To factor an expression is to find two or more expressions whose product is the given expression.

By general agreement the factors must not contain any terms which are under a square root sign, or similar radical signs.

36. A number or an expression is **prime** if it does not have any factors except itself and 1.

Thus, a, 3, $x + y$, $x^2 + y^2$ are all prime numbers.

37. A **monomial factor** of an expression is a number which will exactly divide each term of the expression; as factor **m** below.

For: $$mx + my + mz = m(x + y + z)$$

Similarly,

$$3x^3y + 9x^2y - 3xy = 3xy(x^2 + 3x - 1)$$

38. Complete factoring means that the prime factors of an expression are to be found.

Rule. To find the prime factors of an expression:

1. *First remove any monomial factors of the expression.*
2. *Then factor the resulting polynomial factor, when possible, rewriting all factors which are prime.*
3. *Continue until all factors are prime.*

Example 1. Factor $36ty^8 - 36tz^8$.

Solution. 1. $36ty^8 - 36tz^8 = 36t(y^8 - z^8)$

2. $= 36t(y^4 + z^4)(y^4 - z^4)$

3. $= 36t(y^4 + z^4)(y^2 + z^2)(y^2 - z^2)$

4. $= 36t(y^4 + z^4)(y^2 + z^2)(y + z)(y - z)$

Note 1. When you remove a common numerical factor, be certain that you remove all of it. In this example, it would not be enough to remove 2, 3, 6, 12, or 18.

Note 2. Remember that $y^2 + z^2$ and $y^4 + z^4$ are prime expressions.

Example 2. Factor $2ax^2 - 12ax + 18a$.

Solution. 1. $2ax^2 - 12ax + 18a = 2a(x^2 - 6x + 9)$

2. $= 2a(x - 3)(x - 3)$

EXERCISE 14, *a*

Find the prime factors of:

1. $2\,ax - 6\,ay + 9\,az$
2. $5\,y^3 - 10\,y^2 - 5\,y$
3. $3\,x^2 - 3\,y^2$
4. $am^2 - 10\,am + 24\,a$
5. $12\,a^2b - 16\,ab^2$
6. $cy^2 - 9\,cx^2$
7. $x^4 - y^4$
8. $x^3 - 3\,x^2 - x$
9. $9\,mx^2 - m$
10. $w^2t + 4\,wt - 60\,t$
11. $-\,2a^2 - 6\,a^3 + 8\,a^4$
12. $4\,a^2b - 25\,b$
13. $a^2x - 8\,ax - 33\,x$
14. $-\,5\,x^2 + 10\,x - 5$
15. $x^8 - y^8$
16. $3\,t^2 - 3\,st - 60\,s^2$
17. $a^2y + 7\,aby - 60\,b^2y$
18. $6\,m^2x - 7\,mx + 2\,x$
19. $x^2b + 8\,xb - 33\,b$
20. $a^2m + 17\,am - 38\,m$
21. $3\,a^3 - 108\,a$
22. $x^2z - 2\,xz - 6\,z$
23. $18\,bc^2 - 2\,bx^2$
24. $3\,cx^2 + 6\,cx - 9\,c$
25. $mr^2 + 4\,mr - 21\,r$
26. $18\,t + 2\,at + 2\,a^2t$
27. $24\,k^2 - 18\,k - 15$
28. $30\,a^2b - 120\,ab + 120\,b$
29. $15\,h^2x - 16\,hx - 15\,x$
30. $35\,ct^2 + ct - 12\,c$
31. $15\,x^3d - 10\,x^2d - 25\,xd$
32. $3\,c^2 + 132\,c - 135$
33. $45\,r^4s - 80\,r^2s^2$
34. $3\,k^2 - 33\,k + 72$
35. $-\,x^2 + 25\,x - 100$
36. $-\,a^4 + 18\,a^2 - 77$

EXERCISE 14, *b*. (*Optional*)

1. $x^{2n} - 7\,x^n + 12$
2. $w^{2x} - 10\,w^x + 24$
3. $y^{2a} + 3\,y^a - 40$
4. $x^{2c} + 2\,x^c - 24$
5. $m^{2x} - 4m^x - 21.$
6. $a^{2x} - 8\,a^x - 33$
7. $r^{2y} + r^y - 2$
8. $z^{2m} + 2z^m - 63$
9. $x^{2a} - 6\,x^a + 9$
10. $z^{2n} - 2\,z^nx^m + x^{2m}$
11. $16 - d^{4x}$
12. $r^{2c} - t^{2d}$
13. $x^{4m} - 25$
14. $3\,a^{2x} + 7\,a^x + 2$
15. $6\,m^{2y} - 11\,m^y + 3$
16. $2\,x^{2a} - 3\,x^a - 5$
17. $7\,r^{2c} - 4\,r^c - 11$
18. $2\,y^{2n} + y^n - 15$

39. Factoring polynomials of the form $a^n - b^n$ and $a^n + b^n$. By division, as in § 25, it can be proved that:

(*a*) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

(*b*) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.

(*c*) $a^2 + b^2$, $a^4 + b^4$, $a^8 + b^8$, etc., are not exactly divisible by $a + b$ or $a - b$. They are prime number expressions.

In general we have the following rules.

40. Rules. Letting n represent a positive integer:

I. *$a^n + b^n$, when n is odd, is divisible by $a + b$. The quotient is $a^{n-1} - a^{n-2}b^2 + a^{n-3}b^3 \cdot \cdot \cdot \cdot \cdot b^{n-1}$.*

Observe that the signs of the quotient are alternately plus and minus, that of the first term being plus; that the exponent of a starts with 1 less than n in the first term, and decreases by 1 in each succeeding term; that the exponent of b starts with 1 in the second term, and increases by 1 in each succeeding term until it becomes $n - 1$ in the last term.

II. *$a^n + b^n$, when n is 2, 4, 8, 16, etc., is not factorable. When n is 6, 9, 12, etc., $a^n + b^n$ should be factored by Rule I.*

For example: $x^{12} + y^{12} = (x^4)^3 + (y^4)^3$

III. *$a^n - b^n$, when n is odd, is divisible by $a - b$. The quotient is $a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \cdot \cdot \cdot \cdot \cdot b^{n-1}$.*

Observe that the signs are all plus; and that the terms otherwise are the same as described in Rule I.

IV. *$a^n - b^n$, when n is even, should be factored as the difference of two squares. However, it is divisible by $a - b$, and also by $a + b$, the quotient being obtained as in Rules III or I respectively.*

The following two special cases are particularly important.

$$\mathbf{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$$
$$\mathbf{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$$

Example 1. Factor $32x^5 - 243$.

Solution. 1. $32x^5 - 243 = (2x)^5 - 3^5$

2. $= (2x - 3)[(2x)^4 + (2x)^3 \cdot 3 + (2x)^2 \cdot 3^2 + (2x) \cdot 3^3 + 3^4]$

3. $= (2x + 3)(16x^4 + 24x^3 + 36x^2 + 54x + 81)$

Example 2. Factor $x^9 + y^9$ completely.

Solution. 1. $x^9 + y^9 = (x^3)^3 + (y^3)^3$

2. $= (x^3 + y^3)[(x^3)^2 - (x^3)(y^3) + (y^3)^2]$ (Rule I)

3. $= (x + y)(x^2 - xy + y^2)(x^6 - x^3y^3 + y^6)$ (See § 22)

NOTE 1. $x^9 + y^9$ is also divisible by $x + y$ by Rule I. The quotient is $x^8 - x^7y + x^6y^2 - x^5y^3 + x^4y^4 - x^3y^5 + x^2y^6 - xy^7 + y^8$.

NOTE 2. The correctness of the solution can be checked as usual by substituting values for the literal numbers.

EXERCISE 15

Factor the following expressions completely, when possible: Consider Ex. 33 to Ex. 48 optional.

1. $x^3 + y^3$
2. $y^3 - 8$
3. $y^3 + 27$
4. $w^3 - 64$
5. $8x^3 - 1$
6. $27y^3 + 1$
7. $x^3 - \frac{1}{8}$
8. $w^3 + \frac{1}{27}$
9. $\frac{1}{64} - z^3$
10. $w^3 + \frac{1}{125}$
11. $x^6 - 8$
12. $y^6 + 27$
13. $x^3 + 8a^3$
14. $z^3 - 27w^3$
15. $64m^3 + x^3$
16. $27t^3 - s^3$
17. $x^3 - 8y^3$
18. $b^3 + 27c^3$
19. $c^6 + 8d^3$
20. $8x^6 - 1$
21. $8a^3 - 27b^3$
22. $27m^3 + n^3$
23. $x^6 - 125y^3$
24. $64 + m^9$
25. $x^4 - y^4$
26. $a^6 - 64$
27. $1 - 64b^6$
28. $x^6 - y^6$
29. $x^6 + y^6$
30. $x^8 + y^8$
31. $x^8 - y^8$
32. $r^3s^3 - 27t^3$
33. $x^4 - 16$
34. $x^5 - 32$
35. $1 - y^5$
36. $x^{10} - 1$
37. $32a^5 + 1$
38. $x^{10} - y^{10}$
39. $a^7 - b^7$
40. $\frac{1}{8}x^3 - y^6$
41. $x^6 - 64$
42. $\frac{1}{8}x^3 - y^3$
43. $\frac{1}{27}y^3 - \frac{1}{8}z^3$
44. $\frac{1}{32}a^5 - b^5$
45. $x^3 - .125$
46. $y^3 + .008$
47. $x^3 - .027$
48. $x^{12} - y^{12}$

41. Concerning the factor theorem, and solution of equations by factoring.

Some courses of study require the factor theorem; some make it optional. The College Entrance Examination Board does not require it. In this text, it appears on page 164.

Solution of equations by factoring appears on page 120.

These pages can be studied now if the teacher wishes.

EXERCISE 16. CHAPTER MASTERY-TEST

Part a. Elementary Cases

Factor:

1. $4x^2 - 121$
2. $2y^2 + 11y - 21$
3. $9x^2 - 30x + 25$
4. $1 - 144x^4$
5. $x^2 - 2xy - 99y^2$
6. $12z^2 - 11z - 5$
7. $16t^2 - 24t + 9$
8. $15z^2 - 8z - 12$
9. $25x^2 - \frac{1}{9}y^2$
10. $x^2 - \frac{6}{5}x + \frac{9}{25}$
11. $\frac{1}{16}z^2 - \frac{4}{49}$
12. $x^2 - \frac{3}{2}x + \frac{9}{16}$
13. $x^2 + 1.6x + .64$
14. $z^4 - 1.21$
15. $14m^2 + 3m - 2$
16. $x^2 - \frac{5}{4}x + \frac{25}{64}$

Part b. More Difficult Examples

1. $x^{2n} - 8x^n + 16$
2. $9x^{2a} - 16y^{2b}$
3. $x^{2n} + 2x^n - 63$
4. $2x^{2a} + 9x^a - 35$
5. $x^2 + 2xy + y^2 - 16$
6. $y^2 - 6y + 9 - z^2$
7. $9 - a^2 - 2ab - b^2$
8. $16 - x^2 + 4xy - 4y^2$
9. $8x^3 - y^3$
10. $\frac{1}{8} + x^6$
11. $ax - 2ay + bx - 2by$
12. $(x + y)^2 - 2x - 2y + 1$
13. $4x^2 - y^2 + 2y - 1$
14. $3ax - 3ay - cy + cx$
15. $c^6 + \frac{1}{27}x^3$
16. $x^4 - 16y^4$
17. $(x - y)^2 - 2x + 2y - 15$
18. $c^2 - 4d^2 + 4d - 1$
19. $am + an - \frac{1}{2}n - \frac{1}{2}m$
20. $x^{3a} - y^{3b}$

III. FRACTIONS

DIAGNOSTIC TEST 7

Can you pass the following test on fractions?

1. Reduce to lowest terms: (*a*) $\frac{15}{55}$; (*b*) $\frac{x^2 - 25}{3x + 15}$

2. Multiply: (*a*) $\frac{7}{8} \times \frac{12}{15}$; (*b*) $\frac{x + 3}{x^2 - 4} \cdot \frac{2x + 4}{3x + 9}$

3. Divide: (*a*) $\frac{9}{14} \div \frac{18}{21}$; (*b*) $\frac{a^2 - 2a}{a^2 - 4a + 4} \div \frac{a}{a - 2}$

4. Add: (*a*) $\frac{2}{3} + \frac{3}{4}$; (*b*) $\frac{3}{2a} + \frac{5}{3b}$

5. Subtract: (*a*) $\frac{3}{4} - \frac{5}{16}$; (*b*) $\frac{2}{3m} - \frac{3}{2n}$

6. Add: $\frac{x}{x - 4} + \frac{2x}{x + 5}$

7. Subtract: $\frac{2y}{y - 5} - \frac{y}{y + 4}$

8. Simplify: $\frac{2x}{x^2 - 9} - \frac{1}{x - 3} - \frac{x - 5}{x^2 + 5x + 6}$

9. Simplify: (*a*) $\left(1 + \frac{3}{5}\right) \div \left(3 - \frac{6}{10}\right)$;

   (*b*) $\left(\frac{a}{c} + 1\right) \div \left(\frac{a^2}{c} - c\right)$

10. If x increases, what happens to $\frac{5}{x}$?

NOTE. Mark with a cross (X) the examples you could not solve and master the corresponding parts of the following remedial practice. Do your best also to learn to solve the new types of examples which appear in this chapter.

Remedial Instruction and Practice, Pages 30 to 31

42. A **fraction** indicates the quotient of the **numerator** divided by the **denominator**. The numerator and denominator are called the **terms** of the fraction.

43. First fundamental principle of fractions. *The numerator and denominator of a fraction can be divided by the same number without changing the value of the fraction.*

Thus, if both terms of $\frac{4}{6}$ are divided by 2, we get $\frac{2}{3}$, and $\frac{2}{3} = \frac{4}{6}$.

44. Rule. To reduce a fraction to its lowest terms:

1. *Find the prime factors of both terms.*
2. *Divide both terms by all their common factors.*

Thus: $$\frac{a^2 + 3a - 28}{a^2 - 16} = \frac{\overset{1}{\cancel{(a - 4)}}(a + 7)}{\underset{1}{\cancel{(a - 4)}}(a + 4)}, \text{ or } \frac{a + 7}{a + 4}$$

Caution. $\frac{3x}{4x} = \frac{3}{4}$, since x is a factor and can be removed.

But $\frac{x + 3}{x + 4}$ *cannot be reduced*, as x is *not* a factor. x is an addend and cannot be removed from both terms without changing the value of the fraction. Thus $\frac{2 + 3}{2 + 4}$ does not $= \frac{3}{4}$.

45. Multiplication and division of fractions.

Rule. To find the product of two or more fractions:

1. *Find the prime factors of the numerators and denominators.*
2. *Divide out (cancel) factors common to a numerator and a denominator.*
3. *Multiply the remaining factors of the numerators for the numerator of the product, and of the denominators for the denominator of the product.*

Rule. To divide one fraction by another:

1. *Invert the divisor fraction.*
2. *Multiply the dividend by the inverted divisor.*

EXERCISE 17. REMEDIAL PRACTICE

Reduce to lowest terms:

1. $\frac{5x^3y^4}{3xy^5}$

2. $\frac{12m^2n^3}{20m^3n^2}$

3. $\frac{2(a+b)}{a^2-b^2}$

4. $\frac{3x-3y}{x^2-y^2}$

5. $\frac{a^2-b^2}{(a-b)^2}$

6. $\frac{3r+12}{r^2-16}$

7. $\frac{x^2-y^2}{x^3-y^3}$

8. $\frac{4x+4y}{x^3+y^3}$

9. $\frac{m^2+m-56}{m^2-m-42}$

10. $\frac{y^2-9y+18}{y^2+y-12}$

11. $\frac{3m^2-3n^2}{3m^2+6mn+3n^2}$

12. $\frac{cr^2-cr-12c}{3r^2+13r+12}$

Perform the indicated operations:

13. $\frac{x^2-a^2}{(x+a)^2}\cdot\frac{2x+2a}{3x}$

14. $\frac{4m^2-1}{m^2-16}\cdot\frac{m+4}{2m+1}$

15. $\frac{c^4-d^4}{(c-d)^2}\div\frac{c^2+d^2}{c-d}$

16. $\frac{xy^2-y^3}{x^3+x^2y}\cdot\frac{x^2-xy-2y^2}{x^2-2xy+y^2}$

17. $\frac{3t}{y^2-6y+8}\div\frac{2t}{y^2-y-12}$

18. $\frac{r^3-s^3}{r-s}\div\frac{2r^2+2rs+2s^2}{2r+2s}$

19. $\frac{x^2-1}{2x-4}\cdot\frac{x^2-4}{x^2-x-2}\cdot\frac{3x-6}{x^2+x-2}$

20. $\frac{m^4-1}{16m^4-9n^2}\cdot\frac{4m^2-3n}{2m^2+2}\div\frac{m-1}{4m^2+3n}$

21. $\frac{a^3+b^3}{a^2+3ab+2b^2}\cdot\frac{3a-6b}{3a^2-3ab+3b^2}\div\frac{a^2-4b^2}{a+2b}$

22. $\frac{a^2+7ab+10b^2}{a^2+6ab+5b^2}\cdot\frac{a+b}{a^2+4ab+4b^2}\div\frac{1}{a+2b}$

23. $\frac{x^2+3x+9}{x^2+x-12}\cdot\frac{x^2+2x-8}{x^3-27}\div\frac{x^2-4}{x^2-6x+9}$

24. $\frac{2x^2+7x-15}{2x^2-3x-14}\cdot\frac{2x^2-19x+42}{2x-3}\div\frac{x^2-x-30}{x+2}$

A New Problem in Fractions, Pages 32 to 35

46. Three signs of any fraction must be considered, the sign of the numerator, of the denominator, and of the fraction itself. Certain rules for making changes in these signs follow:

Rule 1. *If the sign of either the numerator or the denominator of a fraction is changed, the sign of the fraction also must be changed.*

Thus: (*a*) $\frac{-4}{+12} = -\frac{1}{3}$; but $\frac{+4}{+12} = +\frac{1}{3}$.

In this example, when -4 is changed to $+4$, the value of the fraction changes from $-\frac{1}{3}$ to $+\frac{1}{3}$.

(*b*) $\frac{-4}{+12} = -\frac{1}{3}$; but $\frac{-4}{-12} = +\frac{1}{3}$.

In this example, when $+12$ is changed to -12, the value of the fraction changes from $-\frac{1}{3}$ to $+\frac{1}{3}$.

Similarly, $\frac{+5}{-20} = -\frac{5}{20}$, or $-.25$; $\frac{-3}{+10} = +\frac{3}{10}$, or $.3$.

Example. Reduce to lowest terms

$$\frac{x^2 - 9}{12 + 2x - 2x^2}.$$

Solution. 1. The terms of the denominator, not being in the same order as those of the numerator, must be rearranged.

Then $$\frac{x^2 - 9}{12 + 2x - 2x^2} = \frac{x^2 - 9}{-2x^2 + 2x + 12}.$$

2. **The negative coefficient of x^2 is inconvenient.** Change the signs of the denominator by multiplying by -1. Then by Rule 1,

$$\frac{x^2 - 9}{-2x^2 + 2x + 12} = -\frac{x^2 - 9}{2x^2 - 2x - 12}$$

3. $$= -\frac{(x-3)(x+3)}{2(x^2 - x - 6)}$$

4. $$= -\frac{\overset{1}{\cancel{(x-3)}}(x+3)}{2\underset{1}{\cancel{(x-3)}}(x+2)}, \text{ or } -\frac{x+3}{2(x+2)}.$$

Rule 2. *If the signs of both the numerator and the denominator of a fraction are changed, the sign of the fraction is not changed.*

Thus, $\frac{-4}{+12} = -\frac{1}{3}$; and also $\frac{+4}{-12} = -\frac{1}{3}$.

In this example, -4 is changed to $+4$, and $+12$ to -12, but the value of the fraction remains $-\frac{1}{3}$.

Similarly, $-\frac{-2x}{-4y} = -\frac{2x}{4y} = -\frac{.5x}{y}$; and $+\frac{-6t}{-2t} = +3$.

47. There are two ways to change the signs of an expression (such as the numerator or the denominator of a fraction).

(*a*) *If the expression is not factored, change the sign of each term of it by multiplying each of them by* -1.

Thus: $(-1)(-2x^2 - 3x + 4) = 2x^2 + 3x - 4$.

(*b*) *If the expression is factored, change the signs in an odd number of factors of the expression.*

Thus: $2 \cdot (-3) \cdot 4 = -24$, but $2 \cdot (+3) \cdot 4 = +24$.

Similarly, consider $(x - y)(y - z)$,

$y - z$ becomes $-y + z$ when its signs are changed.

Therefore $(x - y)(y - z) = -(x - y)(z - y)$, for changing the signs of $y - z$ changes the sign of the product.

On the other hand $(x - y)(y - z) = +(y - x)(z - y)$ because the signs of both $x - y$ and $y - z$ have been changed. This means that -1 has been used twice as a factor, and therefore the sign of the product is not changed.

Do not confuse "changing the signs of the terms of an expression" and "rearranging the terms of an expression."

Thus: $2x - 3$ becomes $3 - 2x$ when its signs are changed.
$2x - 3$ becomes $-3 + 2x$ when its terms are rearranged.

Note. Changing the signs of the terms of an expression changes the value of the expression. Rearranging the terms of an expression does not change the value of the expression.

EXERCISE 18

Simplify the following fractions, expressing the numerical coefficient in decimal form when necessary:

1. $-\dfrac{-2a}{+5a}$

2. $-\dfrac{-3b}{-6b}$

3. $-\dfrac{+c}{-4c}$

4. $+\dfrac{(+2)(-3)}{-12}$

5. $-\dfrac{(-3)(+4)}{-24}$

6. $-\dfrac{(-2)(-12)}{(-15)(+4)}$

7. $+\dfrac{(-7x)(-3y)}{-84xy}$

8. $-\dfrac{5b(-6c)}{(-2c)(-15b)}$

9. $-\dfrac{(-5x^2)(-7y^2)}{(+2y)(-7x^2)}$

10. Reduce to lowest terms $\dfrac{(x-y)(y-z)}{2yz(z-y)}$

Solution. 1. Change the signs of $y - z$. This makes it $z - y$ and changes the sign of the numerator and therefore of the fraction.

2. $\therefore \dfrac{(x-y)(y-z)}{2yz(z-y)} = -\dfrac{(x-y)(z-y)}{2yz(z-y)}$, or $-\dfrac{x-y}{2yz}$

3. This is an inconvenient form of the result. By changing the signs of $x - y$, we change the sign of the fraction.

$$\therefore \quad -\frac{x-y}{2yz} = \frac{y-x}{2yz}$$

$$\therefore \frac{(x-y)(y-z)}{2yz(z-y)} = \frac{y-x}{2yz}.$$

Similarly reduce the following fractions to lowest terms:

11. $\dfrac{a-2b}{4b^2-a^2}$

12. $\dfrac{y^2-z^2}{5z-5y}$

13. $\dfrac{ax-5a}{15-3x}$

14. $\dfrac{m^2-n^2}{(n-m)^2}$

15. $\dfrac{c^2-d^2}{d^3-c^3}$

16. $\dfrac{(a-b)^2}{(b-a)^3}$

17. $\dfrac{(m-n)(n-x)}{(x-m)(n-m)}$

18. $\dfrac{2r^2-2s^2}{2r(s-r)^2}$

19. $\dfrac{3x-15}{20-4x}$

20. $\dfrac{3st-rt}{r^2-9s^2}$

21. $\dfrac{2+t-t^2}{t^2+t-6}$

22. $\dfrac{6m-10}{10-m-3m^2}$

23. $\dfrac{5m-15}{3+2m-m^2}$

24. $\dfrac{4a-3ax}{3x^2+2x-8}$

25. $\dfrac{r^3+s^3}{r^2-s^2}$

48. Changes in signs in multiplication and division.

Example. Simplify $\dfrac{x^2-1}{6+x-x^2} \div \dfrac{1-x}{x-3}$.

Solution. 1. $\dfrac{x^2-1}{6+x-x^2} \div \dfrac{1-x}{x-3} = \dfrac{(x+1)(x-1)}{(3-x)(2+x)} \cdot \dfrac{x-3}{1-x}$

2. $= \left[-\dfrac{(x+1)(x-1)}{(x-3)(x+2)}\right] \cdot \left[-\dfrac{x-3}{x-1}\right]$, or $\dfrac{x+1}{x+2}$

Check. When $x = 2$, $\dfrac{x^2-1}{6+x-x^2} \div \dfrac{1-x}{x-3}$

$= \dfrac{3}{4} \div \dfrac{-1}{-1}$, or $\dfrac{3}{4}$.

Also $\dfrac{x+1}{x+2} = \dfrac{2+1}{2+2}$

$= \dfrac{3}{4}$

EXERCISE 19

Simplify:

1. $\dfrac{x^2+2x+1}{x^2-1} \cdot \dfrac{1-x}{1+x}$

2. $\dfrac{m^2-2m+1}{m^2-1} \div \dfrac{1-m}{m+2}$

3. $\dfrac{x^2-y^2}{y^2-2xy+x^2} \cdot \dfrac{y-2x}{y+x}$

4. $\dfrac{a^2-4a+4}{6a-15} \div \dfrac{4-a^2}{10-4a}$

5. $\dfrac{4-x^2}{6+x-x^2} \cdot \dfrac{x-3}{x-2}$

6. $\dfrac{1+a^3}{2-a} \div \dfrac{1-a^2}{a^2-3a+2}$

7. $\dfrac{x-3a}{4a-x} \div \dfrac{3a-x}{x^2-7ax+12a^2}$

8. $\dfrac{13x-2x^2-15}{9-4x^2} \div \dfrac{x-5}{2x-3}$

9. $\dfrac{(1-a)^3}{a^2-4} \cdot \dfrac{2-a}{(a-1)^2}$

10. $\dfrac{r-2}{6r-3} \div \dfrac{4-r^2}{2-3r-2r^2}$

11. $\dfrac{c^2-2c+1}{1-c^2} \div \dfrac{c^2-8c+7}{7+6c-c^2}$

12. $\dfrac{(x-y)(y-z)}{y-x} \div \dfrac{(y-z)(x-z)}{z-x}$

13. $\dfrac{a-b}{(b-c)(a-c)} \cdot \dfrac{(c-b)(c-a)}{b-a}$

14. $\dfrac{6-3m}{5-10m} \cdot \dfrac{6-2m}{6m+3} \div \dfrac{2m^2-8}{4m^2-1}$

15. $\dfrac{a-2a^2}{4a^2-1} \cdot \dfrac{1+4a+4a^2}{3a} \div \dfrac{2+2a-4a^2}{6a-6a^2}$

Remedial Instruction and Practice, Pages 36 to 38

49. Addition and Subtraction of Fractions.

To add fractions which do not have a common denominator we first change them to equal fractions with a common denominator.

Thus: $$\frac{2}{3}+\frac{1}{2}-\frac{1}{4}=\frac{8}{12}+\frac{6}{12}-\frac{3}{12}, \text{ or } \frac{11}{12}$$

12 is the **lowest common denominator** (L. C. D.) of the fractions.

The change from $\frac{2}{3}$ to $\frac{8}{12}$ is based on the following principle.

50. The second fundamental principle of fractions. *Both terms of a fraction can be multiplied by the same number without changing the value of the fraction.*

51. Combining two or more fractions.

Example 1. Find $\frac{x}{6}+\frac{y}{8}-\frac{z}{3}$.

Solution. 1. $6 = 2 \cdot 3;\ 8 = 2^3;\ 3 = 3$

$\therefore$ the L. C. D. of 6, 8, and 3 $= 2^3 \cdot 3$, or 24.

[The manner of forming the L. C. D. is described in Rule 1, *b*, page 37.]

2. $24 \div 6 = 4$ $\quad \therefore \frac{x}{6}=\frac{4 \cdot x}{4 \cdot 6}, \text{ or } \frac{4x}{24}$

$24 \div 8 = 3$ $\quad \therefore \frac{y}{8}=\frac{3 \cdot y}{3 \cdot 8}, \text{ or } \frac{3y}{24}$

$24 \div 3 = 8$ $\quad \therefore \frac{z}{3}=\frac{8 \cdot z}{8 \cdot 3}, \text{ or } \frac{8z}{24}$

This step should be done mentally.

3. $$\therefore \frac{x}{6}+\frac{y}{8}-\frac{z}{3}=\frac{4x}{24}+\frac{3y}{24}-\frac{8z}{24}, \text{ or } \frac{4x+3y-8z}{24}.$$

Example 2. Find $\frac{a}{x^2}+\frac{b}{xy}-\frac{c}{y^2}$.

Solution. 1. The L. C. D. is x^2y^2. [See Rule 1, *b*, page 37.]

2. $$x^2y^2 \div x^2 = y^2;\ \frac{a}{x^2}=\frac{y^2 \cdot a}{y^2 \cdot x^2}, \text{ or } \frac{ay^2}{x^2y^2};\ \text{etc.}$$

3. $$\frac{a}{x^2}+\frac{b}{xy}-\frac{c}{y^2}=\frac{ay^2}{x^2y^2}+\frac{bxy}{x^2y^2}-\frac{cx^2}{x^2y^2}, \text{ or } \frac{ay^2+bxy-cx^2}{x^2y^2}.$$

52. Rule for adding and subtracting fractions.

1. *Change the fractions, if necessary, to equal fractions which have their lowest common denominator. To do this:*

(*a*) *Factor completely each denominator.*

(*b*) *For the L. C. D., form the product of each of the prime factors which appears in the denominators, using with each the largest exponent it has in any denominator.*

(*c*) *For each fraction: divide the L. C. D. by the denominator of the fraction; multiply both terms of the fraction by the quotient.*

2. *For the numerator of the result, combine the numerators obtained in part c of Step 1 in parentheses, preceding each by the sign of its fraction.*

3. *For the denominator of the result, use the L. C. D.*

4. *Simplify the numerator; reduce the fraction to lowest terms.*

Example. Simplify $\dfrac{6}{x^2 - 6x + 8} - \dfrac{5}{x^2 - 16}$

Solution. 1. $$\frac{6}{x^2 - 6x + 8} - \frac{5}{x^2 - 16}$$

$$= \frac{6}{(x-4)(x-2)} - \frac{5}{(x-4)(x+4)}$$

2. The L. C. D. $= (x-4)(x-2)(x+4)$.

3. L. C. D. $\div (x-4)(x-2) = x + 4$

$$\frac{6}{(x-4)(x-2)} = \frac{6(x+4)}{\text{L. C. D.}}$$

L. C. D. $\div (x-4)(x+4) = x - 2$

$$\frac{5}{(x-4)(x+4)} = \frac{5(x-2)}{\text{L. C. D.}}$$

Hereafter do this step mentally as part of Step 4.

4. $$\therefore \frac{6}{x^2 - 6x + 8} - \frac{5}{x^2 - 16}$$

$$= \frac{6(x+4)}{(x-4)(x-2)(x+4)} - \frac{5(x-2)}{(x-4)(x-2)(x+4)}$$

5. $$= \frac{(6x + 24) - (5x - 10)}{(x-4)(x-2)(x+4)}$$

6. $$= \frac{6x + 24 - 5x + 10}{(x-4)(x-2)(x+4)}, \text{ or } \frac{x + 34}{(x-4)(x-2)(x+4)}$$

EXERCISE 20. REMEDIAL PRACTICE

Perform the indicated operations:

1. $\frac{6a}{5} + \frac{a}{3}$
2. $\frac{5r}{4} - \frac{3r}{8}$
3. $\frac{7x}{6a} + \frac{3x}{4a}$
4. $\frac{4y}{15b} - \frac{5y}{6b}$
5. $\frac{4c}{9a} - \frac{5c}{12b}$
6. $\frac{3t}{7} - \frac{5t}{14} + \frac{t}{2}$
7. $\frac{3w}{2x} - \frac{w}{4x} - \frac{2w}{5x}$
8. $\frac{x-1}{2} + \frac{x+1}{4}$
9. $\frac{y+2}{3} - \frac{y-4}{6}$
10. $\frac{2a-3}{8} - \frac{3a-1}{6}$
11. $\frac{2x+1}{3x} - \frac{3x-1}{4x}$
12. $\frac{x-1}{2x} - \frac{x^2-1}{3x^2}$
13. $\frac{x+3y}{x^2y} - \frac{2x-y}{xy^2}$
14. $\frac{2b-c}{bc} - \frac{c+a}{ac}$
15. $\frac{2x-y}{4y} - \frac{x-3y}{6x}$
16. $\frac{x}{2x+2y} + \frac{y}{3x-3y}$
17. $\frac{3b}{3a-4} - \frac{5b}{5a+6}$
18. $\frac{3}{4p-6} - \frac{4}{15p-12}$
19. $\frac{3a}{a^2-9} - \frac{5}{a-3}$
20. $a - 4 - \frac{2+11a}{3a}$
21. $a - \frac{5a-b}{5a+b} + b$
22. $3a - \frac{3-7a}{2a-3} + 1$
23. $\frac{1}{y^2-xy} + \frac{1}{xy-x^2}$
24. $\frac{a}{m^2-4m} - \frac{a}{m^2-16}$
25. $\frac{2-x}{2+x} - \frac{2+x}{2-x}$
26. $\frac{4p^2+1}{4p^2-1} - \frac{2p-1}{2p+1}$
27. $\frac{3}{a-b} - \frac{2b+a}{a^2-b^2}$
28. $\frac{1}{a+b} + \frac{1}{a-b} - \frac{2a}{a^2-b^2}$
29. $\frac{a}{2a-2b} - \frac{b}{3a+3b}$
30. $\frac{r}{r^2-6r+9} - \frac{1}{r^2+4r-21}$
31. $\frac{a+1}{a^2-a-6} - \frac{a-4}{a^2-4a+3} + \frac{a+3}{a^2+a-2}$

NEW TOPICS, PAGES 39 TO 43

53. Combining fractions involving changes in signs.

Example. Simplify $\dfrac{x+2}{x-2}+\dfrac{2-x}{x+2}+\dfrac{16}{4-x^2}$

Solution. 1. Rearrange all expressions in descending powers of x.

2. $\therefore \dfrac{x+2}{x-2}+\dfrac{2-x}{x+2}+\dfrac{16}{4-x^2}$

$= \dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}-\dfrac{16}{x^2-4}$ (By § 46)

3. $= \dfrac{(x+2)(x+2)}{x^2-4}-\dfrac{(x-2)(x-2)}{x^2-4}-\dfrac{16}{x^2-4}$

4. $= \dfrac{(x^2+4x+4)-(x^2-4x+4)-16}{x^2-4}$

5. $= \dfrac{x^2+4x+4-x^2+4x-4-16}{x^2-4}$

6. $= \dfrac{8x-16}{x^2-4}$

7. $= \dfrac{8(x-2)}{x^2-4}$, or $\dfrac{8}{x+2}$

Check: When $x = 1$, $\dfrac{x+2}{x-2}+\dfrac{2-x}{x+2}+\dfrac{16}{4-x^2}$

$= \dfrac{3}{-1}+\dfrac{1}{3}+\dfrac{16}{3}$

$= -3+\frac{1}{3}+5\frac{1}{3}$, or $2\frac{2}{3}$.

Also $\dfrac{8}{x+2}=\dfrac{8}{3}$

$= 2\frac{2}{3}$

EXERCISE 21

1. $\dfrac{1}{x-y}+\dfrac{x}{y^2-x^2}$

2. $\dfrac{1}{3y-y^2}+\dfrac{1}{y^2-9}$

3. $\dfrac{a}{r^2-rs}-\dfrac{a}{s^2-rs}$

4. $\dfrac{1}{2mt-4t}+\dfrac{1}{6t-3mt}$

5. $\dfrac{2}{x-y}+\dfrac{3y+x}{y^2-x^2}$

6. $\dfrac{3}{a+1}+\dfrac{3}{1-a}-\dfrac{6}{a^2-1}$

7. $\dfrac{a}{a+b}-\dfrac{b}{a-b}-\dfrac{b^2}{b^2-a^2}$

8. $\dfrac{r}{r+2}-\dfrac{r}{2-r}-\dfrac{r}{r^2-4}$

9. $\dfrac{2}{r^2-r-6}-\dfrac{3}{9-r^2}$

10. $\dfrac{1}{x^2-4x+4}-\dfrac{2}{6-3x}$

54. A **complex fraction** is a fraction having one or more fractions in either its numerator or denominator, or both. It is merely another way of indicating division.

Example. Simplify $\dfrac{a-\dfrac{b^2}{a}}{a^2-\dfrac{b^4}{a^2}}$.

Solution. 1. $\dfrac{a-\dfrac{b^2}{a}}{a^2-\dfrac{b^4}{a^2}} = \left(a-\dfrac{b^2}{a}\right) \div \left(a^2-\dfrac{b^4}{a^2}\right)$

2. $= \dfrac{\cancel{a^2-b^2}}{\cancel{a}} \cdot \dfrac{\overset{a}{\cancel{a^2}}}{\underset{a^2+b^2}{\cancel{a^4-b^4}}}$, or $\dfrac{a}{a^2+b^2}$

NOTE. A complex fraction can be simplified by multiplying *its* numerator and denominator by the L. C. M. of the denominators of the fractions in its numerator and denominator. In this example, multiply by a^2.

EXERCISE 22, *a*

Simplify:

1. $\dfrac{3}{4-\dfrac{1}{4}}$

2. $\dfrac{2-\dfrac{1}{3}}{3+\dfrac{1}{2}}$

3. $\dfrac{3+\dfrac{2}{5}}{2-\dfrac{3}{10}}$

4. $\dfrac{a-\dfrac{a}{4}}{a+\dfrac{a}{6}}$

5. $\dfrac{m+\dfrac{1}{3}}{m^2-\dfrac{1}{9}}$

6. $\dfrac{3a-\dfrac{1}{2}}{3a+\dfrac{1}{2}}$

7. $\dfrac{\dfrac{2}{x}}{2-\dfrac{1}{x}}$

8. $\dfrac{1+\dfrac{1}{3a}}{a-\dfrac{1}{9a}}$

9. $\dfrac{3t}{4-\dfrac{1}{t}}$

(Continued on page 41.)

10. $\dfrac{y - \dfrac{1}{2}}{y^2 - \dfrac{1}{4}}$

11. $\dfrac{a - \dfrac{b}{2}}{a^2 - \dfrac{b^2}{4}}$

12. $\dfrac{\dfrac{x}{3} - \dfrac{y}{4}}{\dfrac{x}{3} + \dfrac{y}{6}}$

13. $\dfrac{\dfrac{c}{d} + \dfrac{d}{c}}{\dfrac{c}{d} - \dfrac{d}{c}}$

14. $\dfrac{\dfrac{a}{a+b}}{1 - \dfrac{b}{a+b}}$

15. $\dfrac{x - 5 + \dfrac{6}{x}}{x + 1 - \dfrac{6}{x}}$

16. $\dfrac{y - 2 - \dfrac{3}{y}}{y - 1 - \dfrac{6}{y}}$

17. $\dfrac{1 - \dfrac{2s}{r-s}}{\dfrac{r}{3s} - \dfrac{3s}{r}}$

18. $\dfrac{x + \dfrac{2y}{3}}{x^2 - \dfrac{4y^2}{9}}$

19. $\dfrac{1 - \dfrac{1}{x}}{1 - \dfrac{2x-1}{x^2}}$

20. $\dfrac{\dfrac{2a}{a-b} - 1}{\dfrac{a}{a-b} - 1}$

21. $\dfrac{4 - \dfrac{1}{y+1}}{16 + \dfrac{7}{y^2-1}}$

EXERCISE 22, *b* (*Optional*)

Simplify:

1. $\left(\dfrac{2a}{b} - \dfrac{c}{d}\right) \div \left(\dfrac{2a}{b} + \dfrac{c}{d}\right)$

2. $\dfrac{3 + \dfrac{5x-1}{x^2-9}}{3 + \dfrac{2}{x-3}}$

3. $\dfrac{c - \dfrac{2d^2}{c+d}}{c - \dfrac{3cd - d^2}{c+d}}$

4. $\dfrac{a + \dfrac{b^2}{a-b}}{a + \dfrac{2ab + b^2}{a-b}}$

5. $\left(\dfrac{x}{9y} - \dfrac{y}{x}\right) \div \left(\dfrac{1}{6y} + \dfrac{1}{2x}\right)$

6. $\dfrac{x + \dfrac{y^2}{x-2y}}{1 + \dfrac{y}{x-2y}}$

7. $\dfrac{1 + \dfrac{2x+2}{x^2-1}}{2 - \dfrac{x-4}{2x-2}}$

8. $\dfrac{\dfrac{y}{y+2} - \dfrac{y}{y-3}}{\dfrac{y}{y+2} + \dfrac{y}{y-3}}$

55. A **mixed expression** consists of one or more integral terms plus or minus one or more fractions. To simplify such expressions, follow the rule on Page 14.

Example. Simplify $\left(a - \frac{1}{a}\right) \div \left(a - 2 + \frac{1}{a}\right)$

Solution. 1. $\left(a - \frac{1}{a}\right) \div \left(a - 2 + \frac{1}{a}\right) = \left(\frac{a^2 - 1}{a}\right) \div \frac{a^2 - 2a + 1}{a}$

2. $= \frac{(a-1)(a+1)}{a} \cdot \frac{a}{(a-1)(a-1)}$

3. $= \frac{a+1}{a-1}$

EXERCISE 23

Simplify by performing the indicated operations:

1. $\left(a + \frac{b^2}{a}\right) \cdot \left(a - \frac{b^2}{a}\right)$

2. $\left(\frac{x}{9y} - \frac{y}{x}\right) \cdot \left(\frac{6x^2y^2}{x + 3y}\right)$

3. $\left(2 - \frac{x+6}{x-2}\right) \div \left(3 + \frac{5}{x-2}\right)$

4. $\left(\frac{a}{b} + 2 + \frac{b}{a}\right) \div \left(\frac{1}{a} + \frac{1}{b}\right)$

5. $\left(r - \frac{1}{2s}\right) \div \left(4r^2 - \frac{1}{s^2}\right)$

6. $\left(c - \frac{1}{c}\right) \div \left(c - 2 + \frac{1}{c}\right)$

7. $\left(6 + \frac{5x+2}{x^2-1}\right) \div \left(2 + \frac{1}{x-1}\right)$

8. $\left(1 + \frac{5a-4}{a^2-3a-4}\right) \cdot \left(1 - \frac{3a+8}{a^2+a}\right) \div \left(1 + \frac{5a+8}{a^2+a}\right)$

9. $\left(x^2 - 2 + \frac{1}{x}\right) \cdot \left(\frac{x-1}{x^2+x}\right) \div \left(1 - \frac{1}{x^2}\right)$

10. $\left(x + \frac{1}{x^2}\right) \cdot \left(\frac{2}{x+1}\right) \div \left(x - 1 + \frac{1}{x}\right)$

11. $\left(1 + \frac{a^3}{1-a^3}\right) \cdot \left(2a + \frac{2}{1+a}\right) \div \left(\frac{2}{1-a^2}\right)$

12. $\left(1 + \frac{2y^3}{x^3 - y^3}\right)\left(1 + \frac{2xy}{x^2 - xy + y^2}\right) \div \frac{(x+y)^2}{x^2 - y^2}$

Note. Additional Examples appear on p. 239.

56. Dependence of a fraction on its terms.

When the terms of a fraction are changed, then, of course, the value of the fraction changes.

Example. Consider the fraction $\frac{1}{x}$ and let x have in order the values 2, 3, 4, 5, etc. Thus we get the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$. Clearly as x *increases*, $\frac{1}{x}$ *decreases.*

EXERCISE 24

1. If the denominator has a fixed value, and the numerator increases, then the value of the fraction ________.

2. In the fraction $\frac{1}{x}$, if x is doubled then $\frac{1}{x}$ is ________.

3. In the fraction $\frac{y}{2}$, if y is trebled, then $\frac{y}{2}$ is ________.

4. If x increases, then $1 + \frac{1}{x}$ ________.

5. If y decreases, then $1 - \frac{1}{y}$ ________.

6. If x increases, then $\frac{1 + x}{1 - x}$ ________.

7. The average of two numbers is $\frac{a + b}{2}$. If a increases while b remains fixed in value then the average ________.

8. $b = \frac{A}{h}$ is a formula from geometry. If A has a fixed value, and h increases then b ________; if h is doubled then b is ________.

9. $C = \frac{E}{r + R}$ is a formula met in the study of electricity. If E and r have fixed values, and R increases, then C ________.

10. If x increases, then $-\frac{1}{x}$ ________.

EXERCISE 25. CHAPTER MASTERY-TEST

1. Reduce $\dfrac{x^2 + 3x + 2}{x^2 - 4}$

2. Does $\dfrac{2x + 1}{2x + 2} = \dfrac{1}{2}$?

3. Reduce $\dfrac{x^3 - y^3}{y^2 - 2xy + x^2}$

4. Reduce $\dfrac{-a(c - b)}{b(b - c)}$

5. Find $\dfrac{m}{5x - 15} + \dfrac{m}{7x + 21}$

6. Find $\dfrac{6}{x^2 - 4} - \dfrac{5}{(x - 2)^2}$

7. Find $\dfrac{x - 3}{x^2 - 3x + 2} - \dfrac{x + 1}{2x^2 - 8}$

8. Find $\dfrac{9x^2 - 1}{x^3 - 16x} \cdot \dfrac{x^2 + 4x}{3x + 1}$

9. Find $3m - 4 - \dfrac{9m^2 - 16}{3m + 4}$

10. Find $\dfrac{c^2 - 9}{3c - 6} \cdot \dfrac{c^2 - 4}{c^2 - c - 6} \div \dfrac{c^2 + 2c - 3}{3c - 3}$

11. Find $\left(\dfrac{c}{d} - \dfrac{d}{c}\right) \div \left(1 - \dfrac{d^2}{c^2}\right)$

12. $C = \dfrac{E}{r + \dfrac{R}{n}}.$ Find C if $E = \dfrac{7}{2}$; $r = \dfrac{5}{6}$; $R = \dfrac{1}{8}$; $n = 3$.

13. $h = \dfrac{3V}{b}.$ If V has a fixed value, how does h change when b increases?

14. $c = \dfrac{2A}{h} - b.$ If A and h have fixed values, how does c change when b increases?

15. Simplify $\dfrac{\dfrac{3}{4} - x}{\dfrac{3}{4} + x}$

16. Simplify $\dfrac{x^2 - y^2}{\dfrac{1}{x} - \dfrac{1}{y}}$

17. Simplify $\dfrac{x^2 + 3x + 9}{x^3 - 27} \cdot \dfrac{x^2 - 3x}{6x + 18} \div \dfrac{x^2 - 3x}{3x^2 - 27}$

IV. FIRST DEGREE EQUATIONS IN ONE UNKNOWN

57. The following test will help you discover what you still know about solving simple equations and guide you in the study of this chapter, which is almost entirely review material.

DIAGNOSTIC TEST 8

1. The expression $3x - 5 = 2x + 4$ is an ____________; $3x - 5$ is called its __________ __________; $2x + 4$ is called its __________.

2. If $x - 5 = 6$, then $x =$ ________. This is secured by __________ __________ to both sides.

3. If $x + 5 = 9$, then $x =$ ________. This is secured by __________ __________ from both sides.

4. If $3x = 15$, then $x =$ ________. This is secured by ________ both sides by ________.

5. If $\frac{x}{4} = 3$, then $x =$ ________. This is secured by ________ both sides by ________.

6. If $3x - 5 = 2x + 4$, does $x = 9$? (Yes, or No) ____. If x does equal 9, then x is called the __________ of the equation, and is said to __________ the equation.

7. If $2y + 4 = y + 6$, does $y = 3$? (Yes, or No) ____.

8. Solve the equation $4m + 7 = 3m + 2$. Check.

9. Solve the equation $7t + 3 = 5t + 4$. Check.

10. Solve the equation $11x - 7 = 6x + 2$. Check.

11. Solve the equation $\frac{x}{3} - \frac{x}{2} = \frac{13}{6} - \frac{3x}{5}$. Check.

12. Solve the equation $5(x - 2) = 3(x - 1) - 7$. Check.

58. An **equation** expresses the equality of two numbers. The two parts are called the **sides** or **members** of the equation.

59. An equation **is satisfied by** a set of values of the literal numbers in it if the two sides become numerically equal when these values are substituted for the literal numbers.

60. An **identity** is an equation in which the two sides may be made to take exactly the same numerical value or the same form by performing the indicated operations.

(*a*) $10 + 2 = 15 + (2 \times 4) - 11$ is a *numerical identity.*

(*b*) $3\,x(a - b) = 3\,ax - 3\,bx$ is an *identity.* It is satisfied by any set of values of a, b, and x.

61. A **conditional equation** is an equation involving one or more literal numbers which is not satisfied by all values of the literal numbers.

Thus, $3\,x - 4 = x + 6$ *is an equality only when* $x = 5$.

The word equation usually refers to a conditional equation.

62. An equation like $3\,x - 5 = \frac{1}{2}\,x + 7$ is a **first degree** or a **simple** equation. Observe that x has exponent 1 and is not in the denominator of a fraction, or under a radical sign.

63. If an equation has only one unknown number, any value of the unknown which satisfies the equation is called a **root** of the equation.

64. The following axioms are used in solving a simple equation.

(*a*) *The same number may be added to both members of an equation without destroying the equality.*

(*b*) *The same number may be subtracted from both members of an equation without destroying the equality.*

(*c*) *Both members of an equation may be multiplied by the same number without destroying the equality.*

(*d*) *Both members of an equation may be divided by the same number without destroying the equality.*

65. In this text, symbols A, S, M, and D are used to abbreviate the explanations of solutions of equations. Thus:

A_5: means "add 5 to both members of the previous equation."

S_{-3n}: means "subtract $-3n$ from both members of the previous equation."

M_{-1}: means "multiply both members of the previous equation by -1."

D_4: means "divide both members of the previous equation by 4."

Example 1. $\frac{2}{3}x - 5 = 3x - \frac{3}{2}$

Solution. 1. To eliminate the fractional coefficients:

M_6: $\overset{2}{\cancel{6}} \cdot \frac{2}{\cancel{3}}x - 6 \cdot 5 = 6 \cdot 3x - \overset{3}{\cancel{6}} \cdot \frac{3}{\cancel{2}}$

2. $\therefore 4x - 30 = 18x - 9$
3. A_{30}: $4x = 18x + 21$
4. S_{18x}: $-14x = 21$
5. D_{-14}: $x = -\frac{21}{14}$, or $-\frac{3}{2}$

Check.

Does $\frac{2}{3}\left(-\frac{3}{2}\right) - 5 = 3\left(-\frac{3}{2}\right) - \frac{3}{2}$?

Does $-1 - 5 = -\frac{9}{2} - \frac{3}{2}$? Yes

EXERCISE 26. REMEDIAL PRACTICE

Solve and check the following equations:

1. $5x - 2 = 2x + 1$

2. $8x - 9 = 2x - 6$

3. $10r - 3 = 5 + 6r$

4. $4y + 3 = 19 - y$

5. $5y - 1 = 4y + 5$

6. $8t - 3 = 25 + 4t$

7. $11b - 6 = 1 - 10b$

8. $21w - 6 = 1 - 7w$

9. $4t - 3 = 3 - t$

10. $15w - 5 = 7 - 9w$

11. $10p - 1 = -5p + 4$

12. $7x - 1 = 3x + .6$

13. $x - \frac{2}{5}x = \frac{9}{5}$

14. $\frac{1}{2}y + \frac{1}{6}y = \frac{10}{3}$

15. $\frac{4}{5}t = \frac{1}{2}t + \frac{9}{5}$

16. $\frac{1}{6}x + \frac{1}{4}x = \frac{7}{3} + \frac{2}{9}x$

17. $\frac{5}{12}c - \frac{13}{5} = \frac{1}{5}c$

18. $\frac{1}{3}m - \frac{1}{4}m = \frac{10}{3} - \frac{1}{18}m$

19. $4(x - 1) - 7 = 9$

20. $5(t + 6) - 10 = 50$

66. There are **three mechanical methods of solving a simple equation** the use of which is optional.

(*a*) **Transposition.** A term may be transposed from one side of an equation to the other, provided its sign is changed.

Proof. 1. Let $x + a = b.$

2. S_a. $x = b - a.$ Axiom (*b*), § 64

The result is the same *as if* $+ a$, *with its sign changed*, were taken from the left side and placed on the right side of the equation.

(*b*) **Cancellation.** A term which appears in both sides of an equation may be cancelled.

Proof. 1. Let $x + a = b + a.$

2. S_a. $x = b.$ Axiom (*b*), § 64

The result is the same *as if* $+ a$ *were simply dropped* from both sides of the equation.

(*c*) **Changing signs in an equation.** The signs of all of the terms of an equation may be changed.

Proof. 1. Let $ax - b = c - dx.$

2. M_{-1}. $-ax + b = -c + dx.$ Axiom (*c*), § 64

The result is the same *as if the signs of all the terms of the equation were simply changed.*

Remember, these rules need not be used. They *are used* commonly by mature mathematicians, so your teacher may wish you to become acquainted with them. Use them or not as your teacher directs.

Example. Solve the equation

$$5 - 6x - 11x = 13 - 11x - 4x$$

Solution. 1. $5 - 6x - 11x = 13 - 11x - 4x$

2. Cancelling $11x$: $5 - 6x = 13 - 4x$

3. Transposing 5 and $-4x$: $-6x + 4x = 13 - 5$

4. Collect terms (C. T.) $-2x = 8$

5. Changing signs: $2x = -8$

6. $x = -4$

67. Solving equations with fractional numerical coefficients.

Example. Solve the equation

$$\frac{5y^2 - 4y}{5} - \frac{17}{2} = \frac{10y^2 + 9y}{10}$$

Solution. 1. First we must eliminate the denominators. We do this by multiplying by the L. C. D., namely 10.

2. M_{10}: $\quad \overset{2}{\cancel{10}} \cdot \frac{5y^2 - 4y}{\cancel{5}} - \overset{5}{\cancel{10}} \cdot \frac{17}{\cancel{2}} = \overset{1}{\cancel{10}} \cdot \frac{10y^2 + 9y}{\cancel{10}}$

3. $\cancel{10y^2} - 8y - 85 = \cancel{10y^2} + 9y$

4. A_{85}; S_{9y} $\quad -17y = 85$

5. D_{-17} $\quad y = -5$

Check. Does $\frac{5 \cdot 25 + 20}{5} - \frac{17}{2} = \frac{10 \cdot 25 - 45}{10}$

Does $\frac{145}{5} - \frac{17}{2} = \frac{205}{10}$? Does $29 - 8\frac{1}{2} = 20\frac{1}{2}$? Yes.

EXERCISE 27. REMEDIAL PRACTICE

Determine by substitution which of the numbers 1, 2, − 3, and $\frac{1}{2}$ are roots of the following equations:

1. $5x - 2 = 2x + 1$
2. $x^2 + 2x = 4$
3. $8x - 9 = 2x - 6$
4. $2x^2 + 3x - 2 = 0$

Solve the following equations:

5. $3(n - 2) = 14 - n$
6. $\frac{1}{3}x = \frac{2}{7}x - 1$
7. $\frac{7}{2}t - \frac{4}{3}t = \frac{11}{6} - \frac{2}{5}t$
8. $\frac{4}{9}y - 1 = \frac{7}{9}y - \frac{16}{3}$
9. $\frac{5}{4}z - \frac{3}{2}z = \frac{1}{8} - \frac{1}{3}z$
10. $4m - \frac{23}{20} = \frac{6}{5}m + \frac{1}{2}m$
11. $\frac{1}{3}(x - 2) = \frac{1}{2}(7 - x)$
12. $\frac{1}{4}(x + 7) = \frac{1}{7}(4 - x)$
13. $\frac{1}{3}(3y - 10) = \frac{1}{5}(10 - 3y)$
14. $\frac{1}{10}(1 - x) = \frac{19}{15} - \frac{1}{3}x$
15. $\frac{1}{3}(x + 5) - 4 = \frac{1}{4}(x - 10)$
16. $\frac{3}{10}(m - 1) - \frac{1}{3}(5m + 7) = \frac{17}{6}$
17. $\frac{1}{2}(s - 3) = \frac{1}{3}s - \frac{1}{2}(3s - 7)$
18. $\frac{3}{2}(y - 1) = \frac{1}{2}(y - 3) - \frac{7}{10}$

68. Solving fractional equations.

Example. Solve the equation

$$\frac{3}{2x+1} - 1 = \frac{1+4x-4x^2}{4x^2-1}$$

Solution. 1. Eliminate the denominators by multiplying by the L. C. D.; that is $M_{(4x^2-1)}$

$$\overset{(2x-1)}{\cancel{(4x^2-1)}}\left(\frac{3}{\cancel{2x+1}}\right) - (4x^2-1) = \cancel{(4x^2-1)}\,\frac{1+4x-4x^2}{\cancel{4x^2-1}}$$

2. $\therefore\ 6x - 3 - 4x^2 + 1 = 1 + 4x - 4x^2$

3. A_{4x^2} $\quad 6x - 2 = 4x + 1$

4. $2x = 3$, or $x = \frac{3}{2}$

Check: Does $\frac{3}{4} - 1 = \frac{1+6-9}{9-1}$? Does $-\frac{1}{4} = \frac{-2}{8}$? Yes.

In the check, observe that $2x + 1 = 2\left(\frac{3}{2}\right) + 1 = 3 + 1$, or 4.

Example 2. Solve the equation

$$\frac{2t+7}{6t-4} - \frac{3t-5}{9t+6} = \frac{17t+7}{9t^2-4}$$

Solution. 1. First factor the denominators:

$$\frac{2t+7}{2(3t-2)} - \frac{3t-5}{3(3t+2)} = \frac{17t+7}{(3t-2)(3t+2)}$$

2. Multiply both sides by $6(3t-2)(3t+2)$. To do this, imagine the multiplier as standing in front of each fraction in turn.

For example, $\overset{3}{\cancel{6}}\cancel{(3t-2)}(3t+2)\cdot\frac{2t+7}{\cancel{2}\cancel{(3t-2)}} = 3(3t+2)(2t+7).$

In this way you will get:

$$3(3t+2)(2t+7) - 2(3t-2)(3t-5) = 6(17t+7)$$

3. $\therefore\ 3(6t^2+25t+14) - 2(9t^2-21t+10) = 102t + 42.$

4. $\therefore\ 18t^2 + 75t + 42 - 18t^2 + 42t - 20 = 102t + 42$

5. $\therefore\ 117t + 22 = 102t + 42$

6. $\therefore\ 15t = 20$, or $t = \frac{3}{4}$

NOTE. If the answer found makes any denominator zero, then it cannot be a root of the given equation for division by zero is not permitted. (See § 20.)

EXERCISE 28. REMEDIAL PRACTICE

Solve and check the following equations:

1. $\frac{s-3}{2s}=\frac{1}{3}-\frac{3s-7}{2s}$

2. $\frac{5c-4}{5}=\frac{10c+9}{10}-\frac{17}{2c}$

3. $\frac{2-x}{5x}=\frac{4}{15x}-\frac{1}{6}$

4. $\frac{x-5}{x+3}=\frac{x-4}{x+6}$

5. $\frac{4}{x+1}=\frac{3x+1}{x^2-1}$

6. $\frac{2}{y-1}=\frac{9-y}{y^2-y}$

7. $\frac{5-x}{x^2-1}=\frac{1}{x-1}$

8. $\frac{7+m}{m-3}=\frac{m+4}{m-5}$

9. $\frac{3a}{a-3}-\frac{8}{5a-15}=1$

10. $\frac{9x+2}{3x-1}=5-\frac{2x-3}{x-1}$

11. $\frac{4}{x-4}-\frac{3}{x-3}=\frac{7}{3x-9}$

12. $\frac{4}{x-5}+\frac{5}{4+x}=0$

13. $\frac{1}{4x}+\frac{1}{2x-2}=\frac{2}{x(x-1)}$

14. $\frac{x+3}{2-x}-\frac{12}{4-x^2}=\frac{3-x}{2+x}$

15. $\frac{z}{z-4}=\frac{z}{3z-12}-2$

16. $\frac{x}{x-6}-\frac{3}{2x-12}=\frac{x+8}{x+2}$

17. $\frac{3x-2}{x+3}-\frac{36-4x}{x^2-9}=\frac{2+3x}{x-3}$

18. $\frac{3}{x-3}-\frac{2}{x^2-5x+6}=\frac{2}{x-2}$

NOTE. See Note, p. 50.

19. $\frac{x}{x+3}-\frac{x}{x+2}=\frac{3x+10}{x^2+5x+6}$

20. $\frac{3}{2x-1}+\frac{5}{4x-2}=\frac{11}{6}$

21. $\frac{x}{x+7}=\frac{x}{x+1}-\frac{x+25}{x^2+8x+7}$

22. $\frac{6x^2+14x-4}{x^2-4}=\frac{x}{x+2}+5$

EXERCISE 29 (*Optional*)

Harder Fractional Equations

Solve the following equations:

1. $\frac{x+5}{x-3}+\frac{4}{x-3}=5$

2. $\frac{3x+4}{2x-3}+3=\frac{3-x}{2x-3}$

3. $\frac{15x^2-5x-8}{3x^2+6x+4}=5$

4. $\frac{3}{r-2}-\frac{4}{2r-1}=\frac{1}{r+4}$

5. $\frac{9}{3c-5}-\frac{2}{c-2}=\frac{1}{c-3}$

6. $\frac{4-x}{1-x}=\frac{12}{3-x}+1$

7. $\frac{2x}{3x-4}=\frac{4x+5}{6x-1}-\frac{3}{3x-4}$

8. $\frac{6x+5}{2x^2-2x}-\frac{2}{1-x^2}=\frac{3x}{x^2-1}$

9. $\frac{3m}{2m-5}-\frac{7}{3m+1}=\frac{3}{2}$

10. $\frac{4x}{2x-6}-\frac{4}{5x-15}=\frac{1}{2}$

11. $\frac{3y}{2y+3}+\frac{2y}{3-2y}=\frac{2y^2-15}{4y^2-9}$

12. $\frac{2y-14}{y^2+3y-28}=\frac{y+3}{y+7}-\frac{y-2}{y-4}$

13. $\frac{a+2}{a-4}-\frac{2a-3}{a+3}=\frac{26-a^2}{a^2-a-12}$

14. $\frac{2z+1}{z-8}-\frac{2z-1}{z+6}=\frac{18z+34}{z^2-2z-48}$

15. $\frac{2x-1}{2}-\frac{x+2}{2x+5}=\frac{6x-5}{6}$

16. $\frac{y}{3}=\frac{9y-2}{9}-\frac{2y^2}{3y-4}$

17. $\frac{2x+7}{14}-\frac{x+6}{7}=\frac{5x-4}{3x+1}$

18. $\frac{5t+1}{8}-\frac{4t}{5t+8}=\frac{3t-2}{3}+\frac{7}{5t+8}$

19. $\frac{3x-2}{x+3}=\frac{36-4x}{x^2-9}-\frac{2+3x}{3-x}$

69. Solving literal equations. An equation like

$$a(bx - a) = b(a - bx)$$

is a literal equation. In such equations, we assume x is the "unknown."

Solution. 1. $$a(bx - a) = b(a - bx)$$

2. $$\therefore abx - a^2 = ab - b^2x$$

3. Get the terms containing x on one side, and the remaining terms on the other side.

$$\therefore abx + b^2x = a^2 + ab$$

4. Factor out the coefficient of x.

$$(ab + b^2)x = a^2 + ab$$

5. $D_{(ab+b^2)}$ $$x = \frac{a^2 + ab}{ab + b^2}$$

6. Reduce to lower terms: $x = \dfrac{a(a + b)}{b(a + b)}$, or $x = \dfrac{a}{b}$

Check. Does $a\left(b \cdot \dfrac{a}{b} - a\right) = b\left(a - b \cdot \dfrac{a}{b}\right)$?

Does $a(a - a) = b(a - a)$? Does $a \cdot 0 = b \cdot 0$? Yes.

EXERCISE 30

Solve the equations:

1. $5(x - 3b) = 3x - 11b$
2. $ax - ac = bx - bc$
3. $r(x - r) = s(s + 2r - x)$
4. $c(x - c^2) = d(d^2 - x)$
5. $ax - a^2 = -b(b + x)$
6. $a(3bx - 2a) = b(2a - 3bx)$
7. $(x - a)^2 + 2x^2 = 3x(x - a)$
8. $m^2(x - 1) - m(x - 2) = 1$
9. $3a^2 - 3ax = 10b^2 - ab - 5bx$
10. $m^3 - nx = mx - n^3$
11. $\dfrac{ax - b}{bx} - 2 = \dfrac{a - b}{abx} - \dfrac{bx + a}{ax}$
12. $\dfrac{x + n}{x - m} - \dfrac{x - n}{x + m} = \dfrac{2(m + n)^2}{x^2 - m^2}$
13. $\dfrac{x - b}{x - 2a} - \dfrac{x + b}{x + 2a} = \dfrac{4a^2}{x^2 - 4a^2}$
14. $\dfrac{2x}{3x - a} - \dfrac{3a}{x + 2a} = \dfrac{2x^2 + ax}{3x^2 + 5ax - 2a^2}$

70. Algebraic translation. Every algebraic expression containing one or more literal numbers is a formula for a number.

Thus, $x^2 - y^2$ is the formula for the difference of the squares of two numbers, although it may appear incomplete. By letting $d = x^2 - y^2$, a formula in customary form is secured. This however is unnecessary.

EXERCISE 31. REMEDIAL PRACTICE

Express in symbols; or, write the formula for:

1. The sum of the cubes of two numbers.

2. The larger part of n if x is the smaller part.

3. The number which is 5 more than x.

4. The number which exceeds y by 3.

5. The result of diminishing t by 8.

6. The excess of 15 over x.

7. The amount by which 18 exceeds twice a given number.

8. One half of the sum of two numbers.

9. (*a*) The distance a man travels in t hours at rate 20 m. p. h. (miles per hour).

(*b*) The distance a man travels in t hours at r m. p. h.

10. The time required to go D miles at the rate r m. p. h.

11. The rate travelled by a party which went D miles in t hours.

12. R per cent of P dollars.

13. The complement of x degrees; the supplement.

14. The number of cents in x nickels and y dimes.

15. The difference between one eighth of a certain number and one twelfth of the same number.

16. The integer (whole number) next larger than the integer n (or *consecutive* to the integer n).

17. The even integer consecutive to (and larger than) the even integer n.

71. Some problems can be solved by expressing the conditions in the **form of an equation.** The ones on the following four pages are commonly required as part of the third semester algebra. *Obviously,* you should read the problem carefully, make certain that you understand all words in it (like *exceeds, consecutive, integer*), understand what you have given and what you are to find.

You must learn how to *translate from English to algebra.* Some of this you have done on page 54. Some problems you can translate quite directly.

Example 1. $\underbrace{\text{Twice a certain number}}$ exceeds 18 by 6

becomes $2n \quad - \quad 18 = 6$

Example 2. Separate 85 into two parts such that the quotient is 2 and the remainder is 5, when the larger is divided by the smaller.

Let the parts be l for the larger and $85 - l$ for the smaller.

$$\frac{l}{85 - l} = 2 + \frac{5}{85 - l}$$

EXERCISE 32. REMEDIAL PRACTICE

1. The denominator of a certain fraction exceeds its numerator by 5. If the numerator be decreased by 3, and the denominator be increased by 1, the resulting fraction has the value $\frac{2}{5}$. Find the fraction.

2. Separate 135 into two parts such that, when the larger is divided by the smaller, the quotient is 3 and the remainder is 23.

3. Find three consecutive integers whose sum is 141.

4. Find three consecutive even integers whose sum is 234.

5. Find three consecutive odd integers such that twice the square of the largest exceeds the sum of the squares of the other two by 208.

72. Uniform motion problems are based on the assumption that some object is moving at the same rate for a specified time.

In such cases, the *distance* = rate × time, or $d = rt$. From this equation, $r = d \div t$ and $t = d \div r$.

Example. An automobile party has 5 hours to spend on a trip. They decide to go as far as they can, traveling at the average rate of 20 miles per hour going, and at 30 miles per hour returning. How far can they go?

Solution. 1. Let d = the no. of miles they can go.

2.

	Distance is	Rate is	Time is
Going	d mi.	20 mi. p. h.	$d \div 20$ hr.
Returning	d mi.	30 mi. p. h.	$d \div 30$ hr.

3. $$\therefore \frac{d}{20} + \frac{d}{30} = 5$$

Solving this equation, $d = 60$ miles.

Check. It takes 3 hr. to go, and 2 to return.

EXERCISE 33. REMEDIAL PRACTICE

1. If two automobiles start toward each other, at the same time, from points 175 miles apart, one going 22 miles per hour and the other 28 miles per hour, how soon will they meet?

2. In Example 1, if the first car starts at 8 A.M. and the second at 9 : 30 A.M., at what time will they meet?

3. A man made a trip of 190 miles in 7 hours, part at the rate of 15 miles per hour and the rest at an average rate of 30 miles per hour. How far did he travel at each rate?

4. The rate of one train is 8 miles per hour less than that of a second train. If the former requires five hours to go the same distance that the latter goes in three and two-thirds hours, what is the rate of each?

Note. Additional problems appear on p. 240.

73. Problems about interest on money. (*Optional*)

EXERCISE 34

1. A man has \$3000 invested at 8% and \$4000 at 7%. How much must he invest at 5% to make his total annual income be 6% of his total investment?

Solution. 1. Let x = the no. of dollars he must invest at 5%.

2.

Am't Invested	At Rate	Interest
3000	8%	240
4000	7%	280
x	5%	$.05\,x$
$(7000 + x)$	6%	$.06(7000 + x)$

3. $$\therefore\ .06(7000 + x) = 240 + 280 + .05\,x.$$

(Complete and check the solution)

2. From three investments, a man receives \$224 interest. The amount invested at 6% is \$1000 more, and the amount invested at 5% is \$400 more than the amount invested at 7%. How much has he invested at each rate?

3. A man has \$1800 invested at 5% and \$2500 invested at 7%. How much must he invest at 8% to make his total income be $6\frac{1}{2}$% of his total investment?

4. A man has \$8000 to invest. He wants to have his total income be 6% on the amount invested. He can invest part at 5% and part at $6\frac{1}{2}$%. What must he invest at each rate?

5. If N dollars are to be invested, part at 8% and part at 5%, how much must be invested at each rate so that the total income will be 6% of the N dollars.

6. A man has A dollars invested at r per cent, and B dollars invested at s per cent. How many dollars must he invest at t per cent to make his total income be n per cent of his total investment?

74. Mixture problems. (*Optional, but required by C. E. E. B.*)

EXERCISE 35

1. How many pounds of 25¢ coffee must be mixed with 40 lb. of 60¢ coffee to make a mixture which can be sold for 40¢?

Solution. 1. Let n = the no. of lb. of 25¢ coffee.

2.

Kind of Coffee	The No. of Lb.	Value
25¢ kind	n	$25\,n$
60¢ kind	40	60×40¢
40¢ kind	$(n + 40)$	$40(n + 40)$¢

3. $$\therefore 25\,n + 60 \times 40 = 40(n + 40)$$
(Complete and check this solution)

2. How many pounds each of 35¢ coffee and 50¢ coffee must a grocer mix to make 100 pounds to sell at 40¢ per pound?

3. How much alcohol must be added to 4 gallons of a 30% mixture of alcohol and water to make a 50% mixture.

Note. This means that 30% of the 4 gallons is alcohol and 70% is water. 50% of the resulting mixture is to be alcohol.

Solution. 1. Let n = the no. of gal. of alcohol to be added.

2.

In the	The Quantity is	The Alcohol is
old solution	4 gal.	$.30 \times 4$ gal.
new solution	$4 + n$	$.50 \times (4 + n)$ gal.

3. $$\therefore .50(4 + n) = n + .30 \times 4$$
(since the alcohol in the new solution consists of that in the old plus the n gallons added. Complete this solution and check it.)

4. A radiator contains $7\frac{1}{2}$ gal. of a 25% mixture of alcohol and water. How much of it must be drained off and replaced by alcohol so that the resulting mixture will be a 50% mixture?

Note. Additional mixture problems appear on p. 241.

V. FUNCTIONAL RELATIONSHIP

75. The **meaning of functional relationship** is easily introduced by some examples.

Examples. (*a*) The total cost of the outfits for a football team depends upon the cost of the outfit for each of the eleven men.

(*b*) The wages earned by a workman who is being paid 75¢ per hour depends upon the number of hours he works.

(*c*) The number of hours required by an automobile for a trip of 150 miles depends upon the average rate per hour.

(*d*) The area of a rectangle depends upon the lengths of its base and its altitude.

Always there is one number or quantity so related to one or more others that any change in the latter causes a change in the former. The numbers or quantities involved in such a situation are said to be **functionally related** to each other. The essential fact is that a definite value of one of the numbers depends upon or corresponds to each value of the other number, or numbers.

Many important functional relationships cannot be expressed by algebraic symbols. Thus, "the price of wheat depends upon the size of the crop." While true, there is not a mathematical formula connecting "price" and "size."

We are concerned especially with those relations which are expressible by algebraic symbols. In this chapter, you will learn some of the means of expressing such relations, and will learn a little about drawing conclusions from such expressions of the relations.

76. One functional relation expressed in four ways.

(*a*) **Expressed in words.** The wages earned by a man working for 75¢ per hour depends upon the number of hours he works.

(*b*) **Expressed by a formula.**

If $n =$ the no. of hr. the man worked,

and $W =$ the no. of dollars he earned,

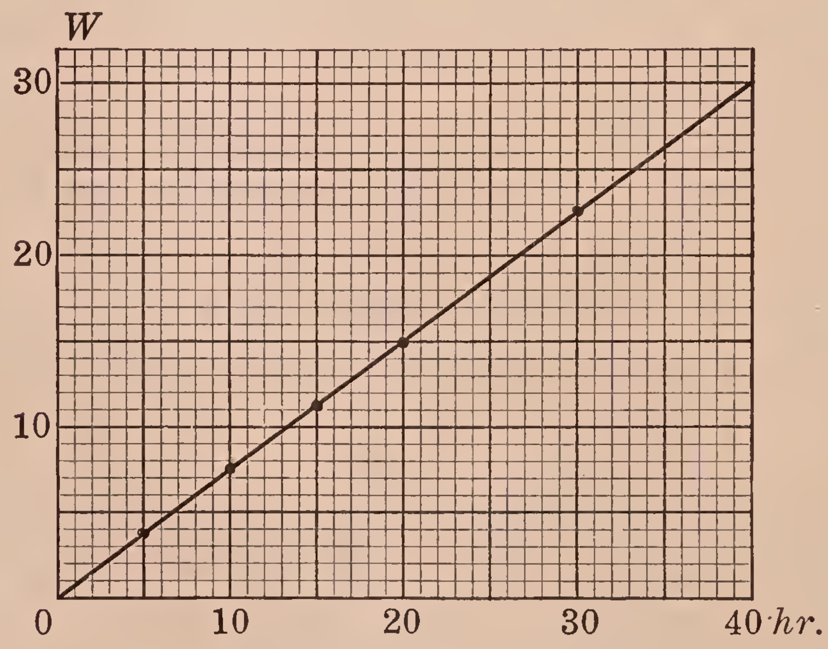
then $h = .75\,n.$

(*c*) **Expressed by a table.**

If the man works	1	2	3	5	10	15	20	30	40 hours
then his wages are	.75	1.50	2.25	3.75	7.50	10.75	15.00	22.50	30.00 dollars

(*d*) **Expressed by a graph.**

On the graph below, the spaces on the horizontal lines represent 1 hour, and those on the vertical lines, 1 dollar.



Each of these means of expressing the relation has some advantage. For this reason it is desirable to know how to express by each of the other three means a relation expressed by one of the four means. In this chapter you will learn how to do this when the relation is expressed first by means (*a*) or (*b*).

77. Some necessary definitions.

Function. If one number is so related to another number (or numbers) that the former has a definite value or values for every value of the latter, then the former is a function of the latter.

Thus, the perimeter of a rectangle is a function of its base and altitude, because the perimeter depends upon them.

78. A **formula** (in a narrow sense) is an algebraic statement of the rule for computing one number which depends upon or *is a function of* one or more other numbers. In a broader sense, a formula is an algebraic statement of the functional relation between two or more numbers.

Thus, $P = 2(a + b)$ expresses the functional relation between P, a, and b.

79. A **variable** is a literal number which has any one of several arithmetical values during a particular discussion.

A **constant** is a number which has some fixed value during a particular discussion.

Thus, if we consider all rectangles of altitude a, then in the relation $P = 2(a + b)$, a is a constant, and P, and b are variables.

The variable (or variables) in a formula which changes first is called the **independent variable.**

The variable whose change is caused by the change in the independent variable (or variables) is called the **dependent variable.**

Thus in the formula $P = 2(a + b)$, if a is constant, and we give values to b, and determine the corresponding values of P, then b is the independent variable and P is the dependent variable. In this case P is a function of b.

If however we give values to P and determine the corresponding values of b, then P is the independent variable and b is the dependent variable. In this case b is a function of P.

80. Write the formulas for the following functional relations.

1. The area (A) of a triangle equals one half the product of its altitude (h) and its base (b).

2. The area of a circle equals π times the square of its radius (r).

3. The area of a regular polygon equals one half the product of its perimeter (p) and the radius (R) of its inscribed circle.

4. The volume (V) of a pyramid equals the product of one third its altitude and the area (B) of its base.

5. The volume of a sphere equals four thirds of the product of π and the cube of its radius.

6. The surface (S) of a sphere equals four times the product of π and the square of its radius.

7. The amount (S) to which P dollars accumulates at $r\%$ compounded annually for n years equals P multiplied by the nth power of the binomial 1 plus r.

8. The sum of the squares of two sides (a and b) of a triangle equals twice the square of one half the third side increased by twice the square of the median drawn to that side.

9. There is a solid whose volume (V) equals one sixth of its altitude (h) multiplied by the sum of its bases (b and c) and 4 times its mid section (m).

10. The number of gallons (g) in a rectangular tank having dimensions L ft., W ft., and H ft. is $\frac{15}{2}$ of the product of L, W, and H.

11. The area of a trapezoid equals one half its altitude multiplied by the sum of its bases.

12. The square upon the hypotenuse (h) of a right triangle equals the sum of the squares of the other two sides (a and b).

13. The force (F) equals the mass (m) times the acceleration (a).

81. Expressing functional relationships by a table.

Example. By the formula $I = prt$, find the values a, b, c and d, in the adjoining table.

Solution (*a*) $I = 1000 \times .05 \times 2 = 50 \times 2$
$= 100$

Solution (*b*) $I = 1500 \times .05 \times 2 = 75 \times 2$
$= 150$

Solution (*c*) $180 = 1500 \times .06\, t;\ 90\, t = 180;$
$t = 2$

Solution (*d*) $270 = P \times .06 \times 3;$
$.18\, P = 270;\ P = 1500$

I	p	r	t
a	$1000	.05	2
b	$1500	.05	2
180	$1500	.06	c
270	d	.06	3

The completed table helps you understand that:

I increases when p increases if r and t are constant;
I increases when r increases if p and t are constant;
I increases when t increases if p and r are constant.

To find the value of any variable in a formula when the values of the other variables are known requires ability to solve *equations*. Examples of this kind will appear as the different kinds of equations are taught.

EXERCISE 36

V	l	w	h
	20	8	5
1000	25	10	
1200	30		4
3600		15	6

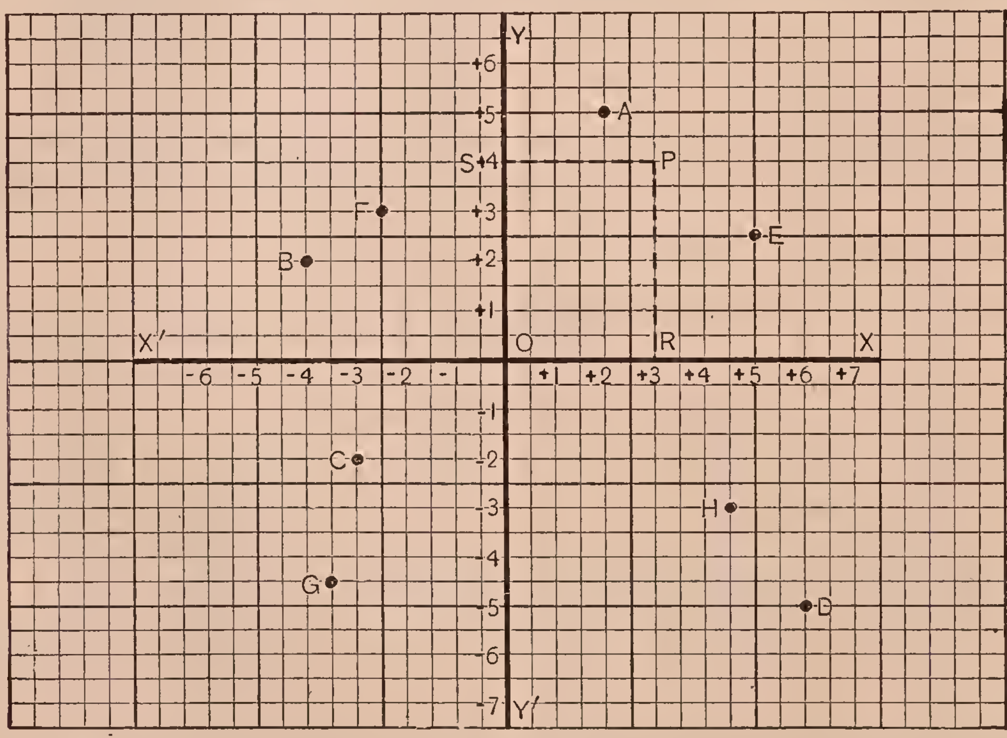
1. Complete the following table using the formula $V = lwh$.
2. V depends upon ________.
3. If l and w are constant, V increases when ______ __________.

C	π	r	h
	$\frac{22}{7}$	14	5
440	$\frac{22}{7}$	14	
396	$\frac{22}{7}$		21

4. Complete the adjoining table using the formula $C = \pi rh$.
5. If r is constant, C increases when __________.
6. If r is constant, and h is doubled, C ________.

82. The vocabulary of graphical representation.

In the figure below: XX' is called the **horizontal axis**; YY' is called the **vertical axis**; together they are called the **axes**; the point O is called the **origin**; PR, perpendicular to the horizontal axis, is called the **ordinate** of the point P; PS, perpendicular to the vertical axis, is called the **abscissa** of P; PR and PS together are called the **coördinates** of P. Distances on OX are considered positive, on OX' negative, on OY positive, and on OY' negative. The part of the plane within the angle XOY is called the *first quadrant;* the part within the angle YOX' is called the *second quadrant;* etc. The abscissa of P, according to the indicated scale, is 3, and the ordinate is 4. The point P is called the **point (3, 4)**.



EXERCISE 37

1. What are the coördinates of A? B? G? H?

2. In what quadrant does a point lie whose coördinates are both negative? Whose abscissa is negative and ordinate positive? Whose abscissa is positive and ordinate negative?

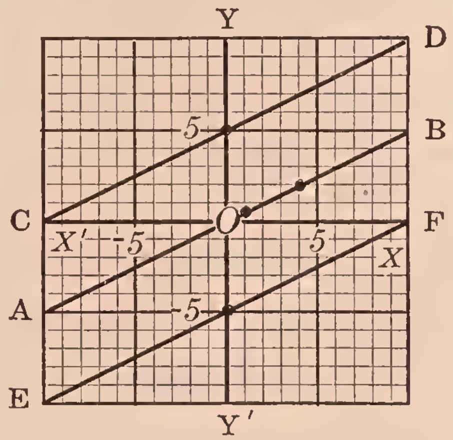
83. Study of first degree functions of x.

Expressions like x, $\frac{1}{2}x$, $2x - 5$ are **first degree functions** of x. We shall study first the functions $\frac{1}{2}x$, $\frac{1}{2}x - 5$, and $\frac{1}{2}x + 5$.

For convenience let $y = \frac{1}{2}x$.

When $x =$	-10,	then $y =$	-5
" $x =$	0,	" $y =$	0
" $x =$	1,	" $y =$	$\frac{1}{2}$
" $x =$	2,	" $y =$	1
" $x =$	4,	" $y =$	2
" $x =$	6,	" $y =$	3
" $x =$	10,	" $y =$	5

The graph is the line AB.



Also let $y = \frac{1}{2}x - 5$

When $x =$	-10,	then $y =$	-10
" $x =$	-4,	" $y =$	-7
" $x =$	0,	" $y =$	-5
" $x =$	$+4$,	" $y =$	-3
" $x =$	$+6$,	" $y =$	-2
" $x =$	$+10$,	" $y =$	0

The graph is the line EF.

Also let $y = \frac{1}{2}x + 5$

When $x =$	-10,	then $y =$	0
" $x =$	-4,	" $y =$	$+3$
" $x =$	0,	" $y =$	$+5$
" $x =$	4,	" $y =$	$+7$
" $x =$	6,	" $y =$	$+8$
" $x =$	$+10$,	" $y =$	10

The graph is the line CD.

Observe that all three functions increase $\frac{1}{2}$ unit when x increases 1 unit. This is caused by the coefficient, $\frac{1}{2}$, of x.

AB goes through the origin and the point $x = 1$, $y = \frac{1}{2}$.

Observe that EF and CD are parallel to AB, and also rise $\frac{1}{2}$ space for each space to the right.

Observe that EF is 5 units below AB; this is caused by the -5 of the function $\frac{1}{2}x - 5$.

The **y-intercept** of $\frac{1}{2}x - 5$ is -5; it is got by letting $x = 0$.

The **x-intercept** is $+10$; it is got by letting $y = 0$.

Observe that CD is 5 units above AB; this is caused by the $+5$ of the function $\frac{1}{2}x + 5$. The *y-intercept* is $+5$; the *x-intercept* is -10.

84. Summary of facts about first degree functions of the form *ax* and *ax* + *b* in which a **and** b **are constants.**

1. *These functions increase* a *units when* x *increases* 1 *unit.*

2. *The graph of every first degree function is a straight line.*

3. *The graph of* ax *goes through the origin; and since the function increases* a *units when* x *increases* 1 *unit, it also goes through the point* $(1, a)$.

4. *The graph of* $ax + b$ *is parallel to* ax*, and is* b *units above or below it according as* b *is positive or negative.*

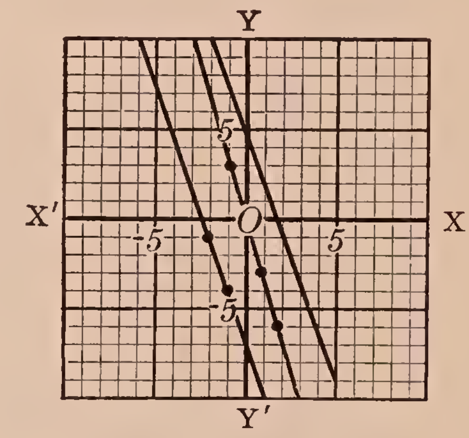
Example 1. What can be said about the functions $y = -3x$, $y = -3x + 5$, and $y = -3x - 7$? Draw their graphs.

Solution. 1. These functions all increase (-3) units when x increases 1 unit; that is they decrease 3 units when x increases 1 unit.

2. $y = -3x$ is the straight line through the origin and the point $1, -3$.

3. $y = -3x + 5$ is parallel to $y = -3x$, and is 5 units above it.

4. $y = -3x - 7$ is parallel to $y = -3x$, and 7 units below it.

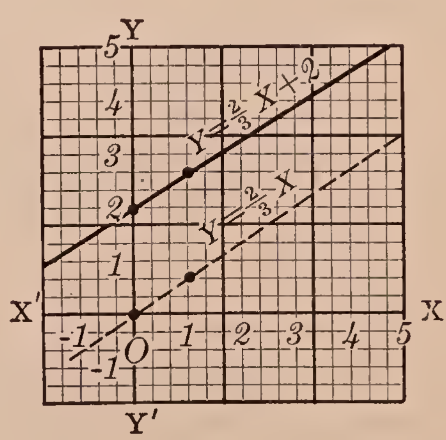


Example 2. Draw the graph of the function $y = \frac{2}{3}x + 2$ and discuss it.

Solution. 1. First draw with a dotted line the graph of $y = \frac{2}{3}x$. This is a straight line through the origin and the point $x = 1$, $y = \frac{2}{3}$.

2. $y = \frac{2}{3}x + 2$ is parallel to $y = \frac{2}{3}x$ and 2 units above it. Draw this line as a full line.

Observe: both lines rise $\frac{2}{3}$ of 1 unit when x increases 1 unit.



For $y = \frac{2}{3}x + 2$, the *y-intercept* is $+2$, for $y = 2$ when $x = 0$.

For $y = \frac{2}{3}x + 2$, the *x-intercept* is -3, for $x = -3$ when $y = 0$.

EXERCISE 38

Make such statements as you can about the following functions; draw the graph and give the x and y intercepts of each.

1. $3x$
2. $-5x$
3. $\frac{1}{3}x$
4. $-\frac{1}{2}x$
5. $4x + 6$
6. $3x - 5$
7. $-2x + 3$
8. $-3x - 2$
9. $\frac{1}{4}x - 5$
10. $-\frac{1}{3}x + 6$
11. $-\frac{1}{4}x - 2$
12. $\frac{1}{3}x + 7$

13. In the formula $A = hb$, let $h = 5$; then $A = 5b$. Now A is a ______ degree function of b. The graph will be a ______ ______ through ______ ______. A will increase ______ units when b ______ 1 unit.

14. In the formula $I = Prt$, let $r = .05$ and $t = 2$. Then $I =$ ______. Now I is a ______ ______ ______ of ______. The graph is a ______ ______ through ______ ______. I ______ ______ units when P increases 1 unit.

15. In $S = C + G$, let $G = 2$. Then S is a ______ ______ function of ______. When C increases 1 unit, then S ______ ______ ______. The graph is a ______ ______. It crosses the C-axis at $C =$ ______. It crosses the S-axis at $S =$ ______.

16. In $d = rt$, let $t = 5$, then $d =$ ______. Now d is a ______ ______ ______ of ______. The graph is a ______ ______ through the ______. For an increase of one unit in r, there is an ______ of ______ units in ______.

17. $F = ma$ is a formula in physics. When $a = 200$, then $F =$ ______. Then F is a ______ ______ function of ______.

18. In $p = 2a + 2b$, let $b = 10$. Then p is a ______ ______ function of ______. When a increases 1 unit, then p ______ ______ unit.

19. In $l = 2a + (n - 1)d$, let $n = 10$ and $d = 2$. Then l is a ______ ______ function of ______. When a decreases 1 unit, then l ______ ______ units.

85. A formula usually indicates definitely how one number can be computed when certain other numbers are given.

86. Some modern views about accuracy in computation.

(*a*) *Degree of accuracy in a measure.* When we say that the radius of a circle is 5, we really mean that it is as nearly 5 as we can determine it. The accuracy of a measure is indicated by the number of significant figures in it; thus 5.1 implies a more accurate measure than 5 or 6, and a less accurate measure than 5.16 or 4.75.

(*b*) *Rounding off numbers.* The value of π is 3.14159.....

3.14159 correct to five figures is 3.1416.

3.1416 correct to four figures is 3.142.

3.142 correct to three figures is 3.14.

When dropping off a final figure 5, computers increase by 1 the preceding figure if it is odd, but do not increase it if it is even.

Thus, 3.135 becomes 3.14, also 3.145 becomes 3.14.

(*c*) *Computing with approximate values.* The circumference of a circle with radius 5 inches is $2 \times 5 \times 3.14159$, or 31.4159 inches.

A reasonable result would be "about 31 inches." Some computers, however, would reason as follows: since there is only one significant figure in 5, then round off the answer to only one significant figure. This gives "about 30 inches."

In the following exercise:

(*a*) Carry out the computation with the numbers given.

(*b*) Give what you consider a "reasonable result."

(*c*) *If your teacher wishes you to secure practice* in use of these new rules of computation, round off your final result so that the number of significant figures is the same as appears in that one of the given numbers which has the least number of significant figures.

EXERCISE 39

1. If $V = 2\pi R(R + H)$, find V when $R = 7$, $H = 13$, and $\pi = 3\frac{1}{7}$.

2. By the formula $T = mg - mf$, find T when $m = 250$, $g = 32.2$ and $f = 75$.

3. By the same formula, find m, when $T = -35{,}600$, $g = 32.2$, and $f = 50$.

4. If $S = \frac{rl - a}{r - l}$, find S, when $r = -2$, $l = 180$, and $a = -20$.

5. By the same formula, find r when $S = 3$, $a = 9$ and $l = 125$.

6. $s = at - \frac{1}{2}gt^2$ is an important formula in physics.
(*a*) Find s when $a = 350$, $t = 4$, and $g = 32.16$.
(*b*) Find a when $s = 543$, $g = 32.16$, and $t = 4$.
(*c*) If t and g have fixed values, how does s change when a decreases?

7. $S = P(1 + r)^n$ is an important formula in the mathematics of finance.
(*a*) Find S when $P = \$5000$, $r = .06$ and $n = 4$. (Note $1.06^4 = 1.26$.)
(*b*) Find P when $S = \$2445$, $r = .05$ and $n = 10$. (Note $1.05^{10} = 1.63$.)
(*c*) If r and n have fixed values, how does S change when P increases?

8. $A = \frac{\pi r^2 E}{180}$ is the formula for the area of a triangle on a spherical surface of radius r.
(*a*) Find A when $r = 10$, and $E = 126°$. (Use $\pi = 3.14$.)
(*b*) Find E when $A = 188$ and $r = 12$. (Use $\pi = 3.14$.)

9. $h = K(1 + \frac{1}{273}t)$ is a formula from the subject of heat in physics. Find h when $K = 91$ and $t = 63$.

87. Finding the value of any number in certain formulas when the values of all the others are given.

Recall that each variable in a formula is a function of all the others. (See § 78.) We are now prepared to find the value of any variable in certain formulas, when the others are given.

Example. If $S = \frac{n}{2}\{2a + (n-1)d\}$, find d when $S = 392$, $a = 2$ and $n = 16$.

Solution. 1. Substituting, $392 = \frac{16}{2}\{4 + 15d\}$

2. $\therefore 392 = 8\{4 + 15d\}$, or $392 = 32 + 120d$

3. $\therefore 360 = 120d$, or $d = 3$

EXERCISE 40. REMEDIAL PRACTICE

In each of the following tables, find the missing value in each line, corresponding to the other values given in that line.

1. $S = \frac{n}{2}\{2a + (n-1)d\}$

S	n	a	d
-99	18	3	
	50	-5	$+3$
442	13		5

2. $V = \frac{h}{6}(a + b + 4m)$

V	h	a	b	m
48	12	5	3	
123	18	10		6
49		10	8	6

3. $T = 2\pi R(R + H)$

T	π	R	H
1760	$3\frac{1}{7}$	14	
	3.14	10	15
954.56	3.14	8	

4. $T = mg - mf$

T	m	g	f
3400	200	32	
4270		32	20
4270		32	25

88. Making a new formula from an old one.

Sometimes it is not as important to have definite values for a particular variable in a formula as it is to have a definite expression of that variable in terms of all the others.

Example. Solve $A = p + prt$, for p.

Solution. 1. $A = p + prt$, or $p + prt = A$

2. Factoring out the coefficient of p

$$p(1 + rt) = A$$

3. $D_{(1+rt)}$ $\therefore p = \dfrac{A}{1 + rt}$

Now we have a direct expression of p in terms of A, r, and t. From it we can tell more easily the effect on p of changes in A, r, or t.

Thus, if r and t are constant, p increases when A increases. If A is constant, p decreases when r, or t, or both increase.

EXERCISE 41

1. Solve $A = p + prt$: (*a*) for r; (*b*) for t.
2. Solve $A = \frac{1}{2} h(l + c)$: (*a*) for h; (*b*) for c.
3. Solve $T = mg - mf$: (*a*) for m; (*b*) for g.
4. Solve $V = \frac{1}{6} h(b + c + 4m)$: (*a*) for h; (*b*) for m.
5. Solve $S = \frac{1}{2} n(a + l)$: (*a*) for n; (*b*) for a.
6. Solve $C = \dfrac{nE}{R + nr}$: (*a*) for E; (*b*) for n.
7. Solve $V = 2\pi R(R + H)$ for H.
8. Solve $S = \dfrac{rl - a}{r - l}$: (*a*) for r; (*b*) for l.
9. Solve $v = \dfrac{MV}{M + m}$: (*a*) for V; (*b*) for M.
10. Solve $s = at - \frac{1}{2} gt^2$ for a.
11. Solve $h = K(1 + \frac{1}{273} t)$ for K.
12. Solve $S = p(1 + r)^n$ for p.
13. Solve $T = 2lh + 2wh + lw$: (*a*) for h; (*b*) for l.

89. Some formulas are made by writing a **general solution** for a problem.

Example. If an agent receives \$2.00 daily and, in addition, a commission of 15% on his sales, then on a day when he sells \$40.00 worth of goods, he receives $\$2.00 + .15 \times 40$, or \$8.00.

We represent his daily income by a formula as follows:

1. Let I = his daily income.
2. Let D = the no. of dollars worth of goods sold.
3. $.15\ D$ = his commission.
4. $I = 2 + .15\ D.$

NOTE. Clearly, the value of I depends on the value of D; as D increases I increases.

EXERCISE 42

1. (*a*) John earns \$25 per week and saves \$4; how much will he have at the end of one year?

(*b*) Write the formula for the amount John will have at the end of 3 years if he earns E dollars per month and spends s dollars.

2. Write the formula for the amount Will earns in n hours at p cents per hour.

3. Write the formula for the total weight (W) of a bus weighing 5000 lb. if it is loaded with n people having an average weight of 140 lb.

4. The parcel post rate for a package going to the third zone is 8¢ for the first pound, and 2¢ more for each additional pound.

(*a*) What is the postage for a package weighing 4 pounds?

(*b*) Write the formula for the postage (p) for a package weighing n pounds.

(*c*) By the formula, find p when $n = 7$.

(*d*) By the formula, find n when $p = 30$.

(*e*) If n increases then p __________,

5. Write the formula for the number of tons of coal in a bin whose dimensions are L, W, and H feet respectively, if one ton occupies 38 cu. ft.

6. The taxi rate in one community is 25¢ for the first one-half mile, and 10¢ additional for every additional one-half mile.

(*a*) What is the charge for a ride of $2\frac{1}{2}$ miles?

(*b*) Write the formula for the charge (c) for a ride of n miles.

(*c*) By your formula, find c when n is 6.

7. The rate for a person to person long distance telephone call from Madison, Wis. to Chicago is $1.00 for the first three minutes, plus 30¢ more for each additional minute.

(*a*) What is the charge for such a conversation lasting 6 minutes?

(*b*) Write the formula for the charge (c) for a conversation lasting n minutes.

(*c*) By your formula, how long may one talk for the sum of $2.50?

(*d*) When n increases then —— ———.

8. The rate for sending a telegram from Madison, Wis. to Minneapolis is 48¢ for the first 10 words, plus $3\frac{1}{2}$¢ for each additional word. Write the formula for the cost (c) of a telegram containing n words.

9. Teachers in a certain community are paid $1250 per year, plus $50 for each additional year of service.

(*a*) Write the formula for the salary (s) of a teacher who is teaching her n'th year.

(*b*) What is the salary for a teacher who is teaching her fifth year in the community?

(*c*) If n increases then s ____________.

10. Write the formula for the average of the marks of a class if a pupils receive the mark 90, b the mark 80, c the mark 70 and d the mark 60.

NOTE. Additional examples appear on p. 242.

EXERCISE 43. CHAPTER MASTERY-TEST

1. One number which ______ __________ one or more other __________ is said to be a __________ of them.

2. A functional relation between two or more numbers can be expressed in __________ ways; namely, (give them).

3. In the functional relation $V = \frac{1}{6} h(b + c + 4m)$, if h is constant then V is a function of __________; then if b, c, and m all increase, V __________. If b, c, and m are constant, then V is a function of h; then if h is doubled V is __________.

4. (*a*) The formula $A = p + prt$ expresses the ______ __________ between __________.

(*b*) Simplify the formula when $r = .05$ and $t = 4$.

(*c*) Using the simplified formula, make a table of corresponding values of A and p for $p = 0, 100, 200$, etc. to 1000 inclusive.

(*d*) Represent the table of part (*c*) by a graph.

5. (*a*) Express in words the functional relation $S = 4\pi r^2$, in which S represents "area of sphere" and r the radius.

(*b*) In this formula, π is a __________; S and r are __________.

(*c*) If r is the independent variable, then S is the ______ __________.

EXERCISE 44. CUMULATIVE REVIEW

1. Divide $x^4 + y^4$ by $x - y$, finding the quotient and remainder. Check your result by using the rule connecting dividend, divisor, quotient, and remainder.

2. Simplify $\dfrac{3x - 6}{x^2 - 4x + 4} \cdot \dfrac{x^2 - 4}{3x - 3} \div \dfrac{5x + 10}{x^3 - 1}$.

3. Combine $\dfrac{s}{s - t} - \dfrac{t}{s + t} - \dfrac{t^2}{s^2 - t^2}$.

4. The quantity $\dfrac{1 - \dfrac{1}{(1 + i)^2}}{i}$ is from the mathematics of finance. Find its value when $i = .05$.

VI. SYSTEMS OF FIRST DEGREE EQUATIONS

90. Can you pass the following test? If you cannot you need the remedial instruction and practice on pages 76 to 84. If you can, you will not require much time for these pages and can go on to the new subject matter on pages 85 to 93.

DIAGNOSTIC TEST 9

1. An equation like $3x + 2y = 12$ has ______ solutions. One such is __________.

2. Two equations like $3x + 2y = 12$ and $x - y = -1$ form a __________ of equations. Usually they have ______ common solution. When they do have, they are called ________ equations.

3. The graph of $3x + 2y = 12$ is __________.
It can be obtained by locating the points
A: $x =$; $y =$; and B: $x =$; $y =$; whose co-ordinates are __________ of the equation.

4. The common solution of $3x + 2y = 12$ and $x - y = -1$ can be obtained __________ or by __________ methods of elimination.

5. The common solution of $3x + 2y = 12$ and $x - y = -1$ obtained graphically is __________.

6. The common solution obtained by the __________ method of elimination is __________.

7. Determine graphically whether $x + y = 5$ and $3x + 3y = 6$ have a common solution. Such equations are called ________ equations.

8. Determine graphically the common solution of $x + y = 5$ and $3x + 3y = 15$. These are __________ equations.

REMEDIAL INSTRUCTION AND PRACTICE, PAGES 76 TO 84

91. Consider the equation $3x - 4y = 12$.

It expresses a functional relation between x and y, because:

(a) for every value of x, there is a value of y.

Thus, when $x = -4$, $y = -6$. $\therefore$ y is a function of x.

(b) for every value of y, there is a value of x.

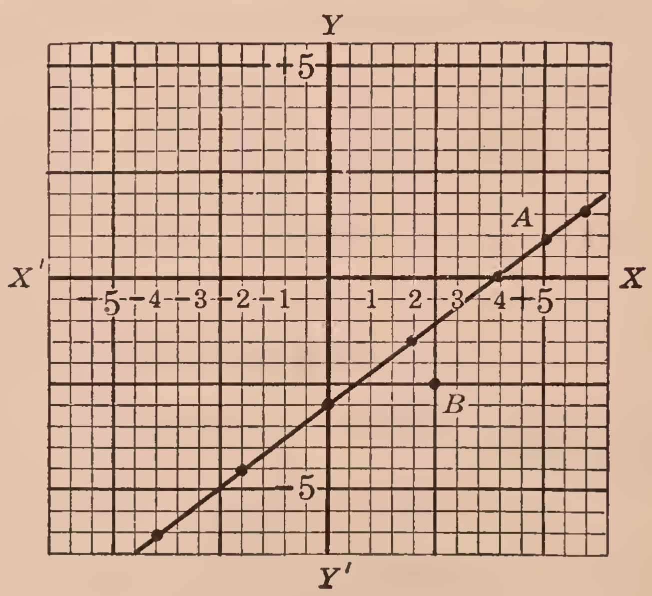
Thus, when $y = 3$, $x = 8$. $\therefore$ x is a function of y.

92. A solution of an equation having two variables consists of a value of one variable and the corresponding value of the other variable, which together satisfy the equation.

By § 91, a, or b, there is an **infinite number** of solutions of an equation like $3x - 4y = 12$.

93. We can represent $3x - 4y = 12$, graphically.

When x is	then y is
-4	-6
-2	$-4\frac{1}{2}$
0	-3
1	$-2\frac{1}{4}$
2	$-1\frac{1}{2}$
4	0
6	$+1\frac{1}{2}$



The graph is a straight line.

Observe point A on the graph. Its coördinates are $x = 5$; $y = \frac{3}{4}$. These satisfy the equation, for $3 \cdot 5 - 4(\frac{3}{4}) = 15 - 3$, or 12.

Observe point B, not on the graph. Its coördinates are $x = 2.5$; $y = -2.5$. These do not satisfy the equation, for $3(2.5) - 4(-2.5) = 7.5 + 10$, or $+17.5$.

94. An equation like $3x - 4y = 12$ is a **rational** and **integral equation of the first degree** having two variables.

It is *rational* since neither x nor y is under a radical sign.

It is *integral* since neither x nor y is in a denominator.

It is of the *first degree* since the sum of the exponents of x and y is one or less in each term of the equation.

95. *The graph of a rational and integral equation of the first degree having two variables is always a straight line.*

Also, every straight line has as equation such a first degree rational and integral equation.

Since every point on the y-axis has 0 as abscissa, the equation of the y-axis is $x = 0$.

Since every point on the x-axis has 0 as ordinate, the equation of the x-axis is $y = 0$.

Since every point on the line parallel to the y-axis through the point $x = 2$, $y = 0$, has abscissa 2, the equation of this line is $x = 2$. Similarly for other lines parallel to the y-axis and lines parallel to the x-axis.

Rule. To obtain the graph of a linear equation.

1. *Find three common solutions of the equation.*

2. *Locate the points whose coördinates are these solutions, and draw the line connecting them.*

If it is a straight line, the work is done correctly.

3. *The x-intercept is the distance from the origin to the point where the line crosses the x-axis. It is found by letting* $y = 0$.

Similarly the y-intercept is found by letting $x = 0$.

EXERCISE 45

Draw the graphs of the following equations.

1. $y = x$	**5.** $x + y = 8$	**9.** $5x + 2y = 20$
2. $y = \frac{1}{2}x$	**6.** $x + 2y = 4$	**10.** $3x = 5 - 2y$
3. $x = 2y$	**7.** $3x - y = 6$	**11.** $y = -2$
4. $x = +4$	**8.** $3x - 2y = 12$	**12.** $x = -3$

96. Two linear equations studied graphically.

Example. What must be the length and the width of a rectangle to have a perimeter of 36 and a length 3 more than twice the width?

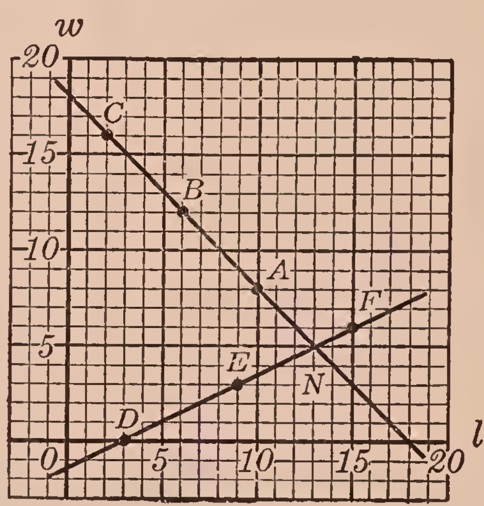
Solution. 1. Let l = the length and w = the width.

2. $\therefore 2\,l + 2\,w = 36$, or $l + w = 18$.
and $l - 2\,w = 3$.

3. Draw the graphs of the equations.

$l + w = 18$			$l - 2\,w = 3$		
Point A:	when $l = 10$,	$w = 8$	Point D:	when $l = 3$,	$w = 0$
" B:	" $l = 6$,	$w = 12$	" E:	" $l = 9$,	$w = 3$
" C:	" $l = 2$,	$w = 16$	" F:	" $l = 15$,	$w = 6$

4. The graphs are at the right.



Line ABC represents graphically the relation between the length and width of a rectangle with perimeter 36.

Line DEF represents graphically the relation between the length and width of a rectangle when the length exceeds twice the width by 3.

5. Observe point N. Its coördinates are 13 and 5. Since it is on both lines, its coördinates satisfy both conditions. That is $l + w = 18$, or $2\,l + 2\,w = 36$, as required. Also $l - 2\,w = 3$, as required.

Remarks. 1. Since there is only one point of intersection of two straight lines, then there is only one common solution for the two equations, and only one solution for the problem.

2. Each equation has solutions which are not solutions of the other. Thus the coördinates of point B satisfy the equation $l + w = 18$ but do not satisfy $l - 2\,w = 3$.

97. Two equations having two variables are **independent** if each has solutions which are not solutions of the other.

Two independent equations which do have common solutions are called **simultaneous equations.**

Two independent equations which do not have a common solution are **inconsistent equations.**

Two equations having two variables are **dependent** if every solution of each is a solution of the other.

98. Two simultaneous linear equations having two variables have **only one common solution.**

Rule. To determine graphically the common solution of two simultaneous linear equations having two variables:

1. *Draw the graphs of the two equations upon one set of axes.*

2. *Determine the coördinates of the point which is on both graphs. This is the common solution. Check the common solution by substituting it in both equations.*

EXERCISE 46. REMEDIAL PRACTICE

Study the following systems of equations graphically. If they are simultaneous, give their common solution.

1. $\begin{cases} x + 4y = 14 \\ 3x + 2y = 12 \end{cases}$

2. $\begin{cases} 2x + y = 7 \\ 2x - y = 5 \end{cases}$

3. $\begin{cases} 3x - 2y = -12 \\ 4x + 2y = -2 \end{cases}$

4. $\begin{cases} 2x + y = 4 \\ 8x + 4y = 16 \end{cases}$

5. $\begin{cases} 5a + 2b = 10 \\ a - b = 9 \end{cases}$

6. $\begin{cases} x + 2y = 3 \\ 2x + 4y = 6 \end{cases}$

7. $\begin{cases} 2x - y = -1 \\ x + y = 7 \end{cases}$

8. $\begin{cases} x - y = 5 \\ x - 2y = 2 \end{cases}$

9. $\begin{cases} x - 3y = 6 \\ 2x - 6y = -1 \end{cases}$

10. $\begin{cases} r - 6s = -10 \\ 2r - 7s = -15 \end{cases}$

11. $\begin{cases} 3x + y = 11 \\ 6x + 2y = -5 \end{cases}$

12. $\begin{cases} 2x - 3y = -14 \\ 3x + 7y = 48 \end{cases}$

13. $\begin{cases} 3x + 2y = 0 \\ 2x + 3y = 5 \end{cases}$

14. $\begin{cases} 4x + 3y = 0 \\ y - x = 7 \end{cases}$

99. The addition or subtraction method of eliminating one variable.

Example. Solve the system $\begin{cases} 5x + 3y = -9 & (1) \\ 3x - 4y = -17 & (2) \end{cases}$

Solution. 1. The coefficients of y are unlike. We can make them alike except for sign by multiplying equation (1) by 4 and equation (2) by 3. Then, if we add, the y terms will drop out.

$M_4(1)$ * Then $20x + 12y = -36$ (3)

2. $M_3(2)$ $9x - 12y = -51$ (4)

3. (3) + (4) $29x = -87$ (5)

4. D_{29} $x = -3$

5. Substitute -3 for x in (1) $-15 + 3y = -9$

6. $3y = 6$, or $y = 2$

7. The solution is $x = -3,\ y = 2$

Check. In (1): Does $5(-3) + 3(2) = -9$?
Does $-15 + 6 = -9$? Yes.

In (2): Does $3(-3) - 4(2) = -17$?
Does $-9 - 8 = -17$? Yes.

NOTE 1. Sometimes one variable has the same coefficient in both equations. In this case subtract instead of adding as in Step 3.

Rule. To eliminate one variable from two equations by addition or subtraction.

1. *If necessary, multiply both equations by such numbers as will make the coefficients of one of the variables of the same absolute value.*

2. *If the coefficients have the same sign, subtract one equation from the other; if they have opposite signs, add the equations.*

3. *Solve the equation found in Step 2 for the other variable.*

4. *Substitute the value of the variable found in Step 3 in any equation containing both variables, and solve for the remaining variable.*

5. *Check by substitution in the original equations.*

NOTE. Remember that this common solution consists of the coordinates of the point on the two graphs.

* NOTE. $M_4(1)$ means "multiply both sides of (1) by 4."

EXERCISE 47. REMEDIAL PRACTICE

Solve the following systems of equations by addition or subtraction. (If difficulty is experienced in obtaining a solution, determine graphically whether the equations are inconsistent or dependent.)

1. $\begin{cases} 8x + 5y = 5 \\ 3x - 2y = 29 \end{cases}$

2. $\begin{cases} 7r - 6s = 63 \\ 9r + 2s = 13 \end{cases}$

3. $\begin{cases} 3c + 7d = -23 \\ 5c + 4d = -23 \end{cases}$

4. $\begin{cases} x - 6y = 2 \\ 2x - 12y = 21 \end{cases}$

5. $\begin{cases} 5x + 3y = -9 \\ 3x - 4y = -17 \end{cases}$

6. $\begin{cases} 5a + 4b = 22 \\ 3a + b = 9 \end{cases}$

7. $\begin{cases} 6x + 2z = -3 \\ 5x - 3z = -6 \end{cases}$

8. $\begin{cases} 8m + 9n = 3 \\ 8m - 9n = 77 \end{cases}$

9. $\begin{cases} 3s + 7t = 4 \\ 7s + 8t = 26 \end{cases}$

10. $\begin{cases} 7c - 2d = 31 \\ 4c - 3d = 27 \end{cases}$

11. $\begin{cases} 5m + 4n = 22 \\ 3m + n = 9 \end{cases}$

12. $\begin{cases} r - 6s = -10 \\ 2r - 7s = -15 \end{cases}$

In Ex. 13 to Ex. 15 simplify by clearing of fractions.

13. $\begin{cases} \dfrac{2x}{3} + \dfrac{3y}{4} = -\dfrac{7}{2} \\ \dfrac{x}{4} - \dfrac{2y}{5} = \dfrac{11}{2} \end{cases}$

14. $\begin{cases} 10a - \dfrac{b-5}{7} = 11 \\ 8b - \dfrac{a+3}{4} = -17 \end{cases}$

15. $\begin{cases} \dfrac{5}{c-2} - \dfrac{7}{3-2d} = 0 \\ \dfrac{1}{2c+5} - \dfrac{8}{7-3d} = 0 \end{cases}$

In Ex. 16 M_3 (1); then M_5(2); then add.

16. $\begin{cases} 2p + \dfrac{5}{q} = -11 \\ 4p - \dfrac{3}{q} = \dfrac{21}{2} \end{cases}$

17. $\begin{cases} \dfrac{3}{r} - \dfrac{1}{s} = 9 \\ \dfrac{4}{r} + \dfrac{3}{s} = -1 \end{cases}$

18. $\begin{cases} \dfrac{10}{x} - \dfrac{9}{y} = 2 \\ \dfrac{8}{x} - \dfrac{15}{y} = -1 \end{cases}$

100. The substitution method of eliminating one variable.

This method can safely be omitted from a minimum course.

Example. Solve the system $\begin{cases} 11x - 5y = 4. & (1) \\ 4x - 3y = 5. & (2) \end{cases}$

Solution. 1. First solve equation (1) for x, as follows:

A_{5y} $\quad 11x = 4 + 5y;\quad D_{11}\quad x = \frac{4 + 5y}{11}$ (3)

2. Substitute this value of x in (2). $\therefore 4\left\{\frac{4+5y}{11}\right\} - 3y = 5$ (4)

3. $\frac{16 + 20y}{11} - 3y = 5$

4. M_{11} $\quad 16 + 20y - 33y = 55$

5. S_{16} $\quad \therefore -13y = 39$, or $y = -3$

6. Substitute -3 for y in (2). $\therefore 4x + 9 = 5$

7. $\therefore 4x = -4$, or $x = -1$.

8. $\therefore$ the common solution is: $x = -1;\ y = -3$.

Check by substituting in equations (1) and (2).

Note 1. You can solve either equation for one variable in terms of the other. Naturally, you select the one which can be solved most easily.

Note 2. Try to do all of Step 1 mentally.

Thus from (2) $x = \frac{3y + 5}{4}$. From (1) $y = \frac{11x - 4}{5}$.

From (2) $y = \frac{4x - 5}{3}$.

Rule. To eliminate one variable from two equations by substitution.

1. *Solve one equation for one variable in terms of the other.*

2. *Substitute for this variable in the other equation the value found for it in Step* 1.

3. *Solve the equation found in Step* 2 *for the second variable.*

4. *Substitute the value of the second variable, obtained in Step* 3, *in any equation containing both variables and solve for the first variable.*

5. *Check by substitution in the original equations.*

EXERCISE 48. REMEDIAL PRACTICE

Solve the following systems by substitution.

(If difficulty is experienced in securing a common solution, determine graphically whether the equations are inconsistent or dependent.)

1. $\begin{cases} 3x + y = 11 \\ 5x - y = 13 \end{cases}$

2. $\begin{cases} 3x + y = 3 \\ 4x + 2y = -2 \end{cases}$

3. $\begin{cases} 2x + 3y = 4 \\ x - 4y = -9 \end{cases}$

4. $\begin{cases} 3x + 4y = 3 \\ 2x - y = 13 \end{cases}$

5. $\begin{cases} x + 5y = 1 \\ 2x + 7y = 1 \end{cases}$

6. $\begin{cases} 4x + 3y = 2 \\ 2x - y = \frac{2}{3} \end{cases}$

First reduce each equation of the following examples to the form $ax + by = c$, which is called the *normal form.*

7. $\begin{cases} 2x - \frac{5}{2}y = 13 \\ \frac{5}{3}x + y = \frac{14}{3} \end{cases}$

8. $\begin{cases} \frac{7}{3}r + 4 = s \\ \dfrac{s}{4} - \dfrac{r}{3} = \dfrac{7}{4} \end{cases}$

9. $\begin{cases} \frac{3}{8}m + n = \frac{1}{4} \\ \frac{1}{2}m - \frac{5}{7}n = \frac{19}{14} \end{cases}$

10. $\begin{cases} \frac{1}{4}a + \frac{1}{5}b = 1 \\ 2a + 18 = b \end{cases}$

11. $\begin{cases} \frac{1}{2}x + \frac{5}{4}y = 2 \\ \frac{1}{6}x - \frac{5}{3}y = \frac{3}{2} \end{cases}$

12. $\begin{cases} \frac{1}{2}x - \frac{1}{3}y = 3 \\ \frac{2}{3}y + \frac{1}{2}x = 0 \end{cases}$

13. $\begin{cases} \dfrac{r + s - 2}{r - s} = -\dfrac{1}{3} \\ \dfrac{3r + s - 3}{2s - r} = -\dfrac{1}{11} \end{cases}$

14. $\begin{cases} \dfrac{y}{3} - \dfrac{x}{2} = 2 \\ \dfrac{3 - 2x}{5} - \dfrac{4 + 5y}{11} = 4 \end{cases}$

15. $\begin{cases} \dfrac{3}{x - 1} + \dfrac{4}{y - 1} = 0 \\ \dfrac{5}{2x - 3} - \dfrac{7}{2y + 13} = 0 \end{cases}$

16. $\begin{cases} \dfrac{r + t}{2} - \dfrac{r - t}{3} = 8 \\ \dfrac{r + t}{3} + \dfrac{r - t}{4} = 11 \end{cases}$

17. $\begin{cases} \dfrac{x - y}{3} - \dfrac{2x + y}{2} = 0 \\ \dfrac{x + 2y}{2} - \dfrac{x}{4} = -\dfrac{11}{4} \end{cases}$

18. $\begin{cases} \dfrac{r + s - 9}{2} = \dfrac{s - r - 6}{3} \\ \dfrac{r - s}{2} = \dfrac{2s}{6} - \dfrac{r + s}{3} \end{cases}$

101. Problems about a number and its digits.

Integers are written by means of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Thus, 38 is a 2 digit number. 3, its tens' digit, represents 30 units, 8 represents units; altogether, *thirty-eight* units. The sum of the digits is 11; their product is 24.

If t is the tens' digit of a number and u is its units' digit, the number contains $(10t + u)$ units.

When the digits are reversed a new number is formed.

Thus reversing the digits of 52 we get 25.

Observe that $52 = (5 \times 10 + 2)$, or *fifty-two* units.
$25 = (2 \times 10 + 5)$, or *twenty-five* units.

EXERCISE 49. REMEDIAL PRACTICE

1. If a certain number of two digits be divided by the sum of its digits, the quotient is 4 and the remainder is 3. If the digits be reversed, the sum of the resulting number and 23 is twice the given number. Find the number.

Solution. 1. Let t = the tens' digit and u = the units' digit.

2. $\therefore 10t + u =$ the given number,
and $10u + t =$ the number with digits reversed.

3. $\therefore 10t + u = 4(t + u) + 3.$ (1)

4. $\therefore 10u + t + 23 = 2(10t + u).$ (2)

(Complete the solution.)

2. The tens' digit of a number of two digits exceeds its units' digit by 4. If the digits be reversed, the new number is 6 more than one half the old number. Find the number.

3. The sum of the two digits of a number is 16; if 18 be subtracted from the original number, the remainder equals the number obtained by reversing the digits. Find the number.

4. If a certain number be divided by the sum of its two digits, the quotient is 7 and the remainder is 6; if the digits be reversed, the sum of the new number and twice the old number is 204. Find the number.

102. Problems about opposing forces.

As you know a river has a current which exerts a *force downstream.* Rowing a boat *downstream* is therefore easier than rowing *upstream.*

If the current flows at the rate of 2 miles per hour, then, theoretically, any one rowing downstream at the rate of 4 miles per hour will proceed at the rate of (4 + 2) or 6 miles per hour; upstream, the rate will be (4 − 2) or 2 miles per hour.

EXERCISE 50

1. If a boat is rowed downstream 10 miles in two hours and the same distance upstream in $3\frac{1}{3}$ hours find the rate of rowing in still water and the rate of the current.

Solution. 1. Let r = the rate of rowing in still water

" c = the rate of the current

$\therefore (r + c)$ = the rate downstream

2. and $(r - c)$ = the rate upstream

3. $\therefore \dfrac{10}{r + c} = 2$ (Since $\dfrac{\text{distance}}{\text{rate}} = \text{time}$)

and $\dfrac{10}{r - c} = 3\frac{1}{3}$

(Complete the solution.)

2. A crew can row 18 miles downstream in 2 hours, and back again in 3 hours. What is the rate of the crew in still water and the rate of the current?

3. An airplane traveled 200 miles with wind of a certain velocity in 4 hours; it returned against a wind of the same velocity in $6\frac{2}{3}$ hours. What was the rate of the wind and of the airplane?

4. A motor boat, which can run at the rate of r miles per hour in still water, went downstream a certain distance in n hours; it took m hours to return.

(*a*) Find the distance and the rate of the current.

(*b*) The distance is a function of ____________.

103. Problems about reciprocals. *(Optional)*

(*a*) The reciprocal of 3 is $1 \div 3$, or $\frac{1}{3}$; of $\frac{3}{2}$ is $1 \div \frac{3}{2}$, or $\frac{2}{3}$.

(*b*) If a piece of work can be done in 5 days, then one fifth of it can be done in 1 day, and n fifths of it in n days.

If a piece of work can be done in x days, then $1 \div x$ part of it can be done in 1 day, $3 \div x$ part in 3 days, and $n \div x$ part in n days.

EXERCISE 51. *(Optional)*

1. A piece of work can be done by A and B working together in 10 days. After working together for 7 days, A stops, and B completes it in 9 days. How long would it take each to do the work alone?

Solution. 1. Let a = the no. of days it would take A alone
and b = the no. of days it would take B alone.

2.

Person	Time Alone	Amount in 1 Da.	Amount in 10 Da.	In 7 Da.	In 9 Da.
A	a Da.	$\frac{1}{a}$	$\frac{10}{a}$	$\frac{7}{a}$	—
B	b Da.	$\frac{1}{b}$	$\frac{10}{b}$	$\frac{7}{b}$	$\frac{9}{b}$

3. $$\therefore \frac{10}{a} + \frac{10}{b} = \frac{ab}{ab}, \text{ or } \frac{10}{a} + \frac{10}{b} = 1$$

$$\text{and } \frac{7}{a} + \frac{7}{b} + \frac{9}{b} = \frac{ab}{ab}, \text{ or } \frac{7}{a} + \frac{16}{b} = 1.$$

(Complete this solution as in Ex. 16, p. 81.)

2. The sum of the reciprocals of certain two numbers is $\frac{5}{2}$. Twice the reciprocal of the larger increased by three times the reciprocal of the smaller is 7. What are the numbers?

3. A and B, working together, did a certain piece of work in 6 days. At another time, they did the same amount when A worked 3 days and B 10 days. How many days would it take for either to do the work alone?

104. Solution of systems of literal equations. (*Optional*)

Problems of this type do not cause any special difficulty. Simply remember and use this fact:

$$ax + bx - cx = (a + b - c)x.$$

Example. Solve the system $\begin{cases} 2\,ax + 2\,by = 4\,a^2 + b^2 & (1) \\ x - 2\,y = 2\,a - b & (2) \end{cases}$

Solution. 1. $M_b(2)$ $\quad bx - 2\,by = 2\,ab - b^2 \quad (3)$

2. Add (1) and (3) $\quad 2\,ax + bx = 4\,a^2 + 2\,ab \quad (4)$

or $x(2\,a + b) = 2\,a(2\,a + b) \quad (5)$

3. $D_{(2a+b)}$ $\quad \therefore x = 2\,a$

4. Substitute in (1) $\quad 4\,a^2 + 2\,by = 4\,a^2 + b^2$

5. $\quad \therefore 2\,by = b^2$, or $y = \dfrac{b}{2}$

Check. Substitute in (2): Does $2\,a - 2\left(\dfrac{b}{2}\right) = 2\,a - b$? Yes.

EXERCISE 52. (*Optional*)

1. $\begin{cases} 2\,ax + y = b \\ ax - 2\,y = c \end{cases}$

2. $\begin{cases} mx + ny = m^2 + 3\,mn \\ 2\,mx - 3\,ny = 2\,m^2 - 4\,mn \end{cases}$

3. $\begin{cases} ax - by = 0 \\ abx - aby = a^2 - b^2 \end{cases}$

4. $\begin{cases} cx + dy = c + d \\ cdx + cdy = c^2 + d^2 \end{cases}$

5. $\begin{cases} ax - by = c \\ bx + ay = d \end{cases}$

6. $\begin{cases} ax - by = a^2 + b^2 \\ ax + by = a^2 + b^2 \end{cases}$

7. $\begin{cases} 2\,bx + ay = a + b \\ 2\,abx + aby = a^2 + b^2 \end{cases}$

8. $\begin{cases} mx + ay = m^2 + n \\ x + amy = m + mn \end{cases}$

9. $\begin{cases} acx = by \\ abcx + aby = (a + b)^2 \end{cases}$

10. $\begin{cases} sy = mx \\ sx + rsy = t \end{cases}$

11. Find two numbers whose sum is s and difference is d.

12. Separate N into two parts whose quotient shall be a and remainder b. (The results are formulas for the two parts.)

13. (*a*) The perimeter of a rectangle is P. Its base exceeds its altitude by d. Find the base and the altitude.

(*b*) Your result shows that b is a function of __________.

105. Systems of equations having three or more variables. When there are two variables, then two equations are necessary to obtain a single common solution. Similarly when there are three variables, then three equations are necessary to obtain a single common solution; four equations, if there are four variables; etc.

However, just as two equations having two unknowns may be dependent, or be inconsistent, so a system of three equations may not have a single common solution. The complete discussion of these facts appears in a later course in algebra.

Example. Solve the system of equations:

$$\begin{cases} 12m - 4n + p = 3 & (1) \\ m - n - 2p = -1 & (2) \\ 5m - 2n = 0 & (3) \end{cases}$$

Solution. 1. $M_2(1)$. $24m - 8n + 2p = 6$ (4)
2. (2) + (4) $25m - 9n = 5$ (5)
3. $M_5(3)$ $25m - 10n = 0$ (6)
4. (5) − (6) $n = 5$
5. Substitute 5 for n in (3). $5m - 10 = 0$, or $m = 2$.
6. Substitute 5 for n and 2 for m in (2).
$2 - 5 - 2p = -1$
7. $\therefore -2p = 2$, or $p = -1$
8. $\therefore m = 2, n = 5, p = -1$

Check. The solution satisfies each of the three given equations.

Rule. To solve a system containing three variables:

1. *From two equations, say the* 1st *and the* 2nd, *eliminate one of the variables; mark the resulting equation, equation* (4).

2. *From another pair of equations, say the* 1st *and* 3rd, *eliminate the same variable; mark the resulting equation, equation* (5).

3. *Solve the system consisting of equations* (4) *and* (5).

4. *Determine the value of the third variable, by substituting in one of the given equations the values found in Step* 3.

5. *Check by substitution in the given equations.*

EXERCISE 53

Solve and check the following systems of equations:

1. $\begin{cases} a - 2b + c = 5 \\ 2a + b - c = -1 \\ 3a + 3b - 2c = -4 \end{cases}$

2. $\begin{cases} 3x + y - z = 11 \\ x + 3y - z = 13 \\ x + y - 3z = 11 \end{cases}$

3. $\begin{cases} 2r + 3s - t = -2 \\ 4r - 3s + 2t = 9 \\ 6r - 6s + 3t = 13 \end{cases}$

4. $\begin{cases} m + 6n + 3p = 8 \\ 3m + 4n = -3 \\ 5m + 7n = 1 \end{cases}$

5. $\begin{cases} 3A - B + 2C = -2 \\ 6A + 3B + 2C = -7 \\ 3A + B - 4C = 5 \end{cases}$

6. $\begin{cases} 2x + 3y - z = -1 \\ 3x - 8z = -1 \\ 5y + 7z = -1 \end{cases}$

7. $\begin{cases} 2x + y - z = 0 \\ 4x + z = 4y \\ y = x + 1 \end{cases}$

8. $\begin{cases} 3x + y - z = 2 \\ y = 1 - 2x \\ 3z = -2y \end{cases}$

9. $\begin{cases} 2x - y + 4z = 0 \\ x - y + 3z = 0 \\ 3x - 2z = \frac{5}{2} \end{cases}$

10. $\begin{cases} x + 2y - z = 7 \\ y + 3x = z \\ 2x + y = -1 \end{cases}$

11. $\begin{cases} \frac{2}{x} - \frac{1}{y} - \frac{3}{z} = 7 \\ \frac{1}{x} + \frac{2}{y} - \frac{1}{z} = 10 \\ \frac{3}{x} - \frac{3}{y} + \frac{2}{z} = -7 \end{cases}$

12. $\begin{cases} \frac{1}{x} - \frac{1}{y} = a \\ \frac{1}{x} - \frac{1}{z} = b \\ \frac{1}{y} + \frac{1}{z} = c \end{cases}$

13. The perimeter of a certain triangle is 100 inches. The sum of the first and second sides exceeds the third side by one half the third side. The sum of the second and third sides equals three sides the first side. What are the lengths of the sides?

14. The sum of the digits of a certain number of three digits is 11. If the number be divided by the sum of its hundreds' and its units' digits, the quotient is 20 and the remainder is 6. The units' digit exceeds the sum of the hundreds' and tens' digits by 1. Find the number.

106. Solving a set of equations by some formulas. (*Optional*)

Consider the set of equations $\begin{cases} ax + by = c. \\ dx + ey = f. \end{cases}$

If solved as in § 103, you will find that: $\quad x = \dfrac{ce - bf}{ae - bd}; \quad y = \dfrac{af - cd}{ae - bd}.$

Some mathematician saw that $ae - bd$ can be obtained by arranging a, b, d, and e thus $\begin{vmatrix} a & b \\ d & e \end{vmatrix}$ and taking the product of the downward diagonal minus the product of the upward diagonal.

.

$\begin{vmatrix} a & b \\ d & e \end{vmatrix}$ is defined as being $ae - bd$. It is called a **determinant.**

Similarly $\quad \begin{vmatrix} c & b \\ f & e \end{vmatrix} = ce - bf; \quad \begin{vmatrix} a & c \\ d & f \end{vmatrix} = af - cd.$

$$\text{Therefore } x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}; \quad y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}.$$

The advantage of these formulas is obvious from the rule.

Rule. To solve a set of two equations by determinants:

1. *Arrange the equations in the form:* $\begin{cases} ax + by = c. \\ dx + ey = f. \end{cases}$

2. *There is a common solution if* $\begin{vmatrix} a & b \\ d & e \end{vmatrix}$ *is not zero.*

3. *The value of x is a fraction; its denominator is the determinant formed by the coefficients of x and y; its numerator is the determinant obtained by replacing the coefficients of x in the denominator determinant by the corresponding absolute terms.*

4. *The value of y is a fraction with the same denominator as x; its numerator is the determinant obtained by replacing the coefficients of y in the denominator determinant by the absolute terms.*

Example 1. Solve $\begin{cases} 2x - 5y = -16 \\ 3x + 7y = 5 \end{cases}$ by determinants.

Solution. 1.

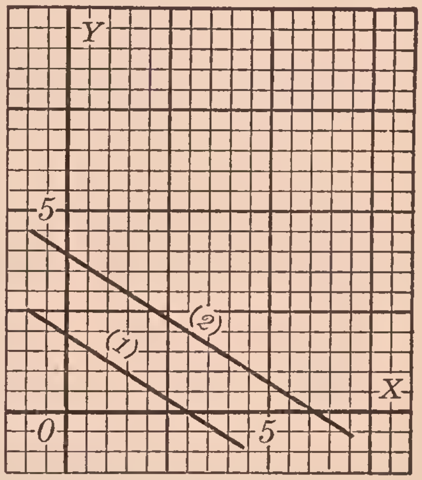
$$x = \frac{\begin{vmatrix} -16 & -5 \\ 5 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 3 & 7 \end{vmatrix}} = \frac{7(-16) - 5(-5)}{2 \cdot 7 - 3(-5)} = \frac{-112 + 25}{14 + 15} = -3$$

$$y = \frac{\begin{vmatrix} 2 & -16 \\ 3 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 3 & 7 \end{vmatrix}} = \frac{2 \cdot 5 - 3(-16)}{2 \cdot 7 - 3(-5)} = \frac{10 + 48}{14 + 15} = \frac{58}{29} = 2$$

Example 2. Consider the system $\begin{cases} 2x + 3y = 6 & (1) \\ 4x + 6y = 24 & (2) \end{cases}$

$$x = \frac{\begin{vmatrix} 6 & 3 \\ 24 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}} = \frac{36 - 72}{12 - 12} = \frac{-36}{0}$$

$$y = \frac{\begin{vmatrix} 2 & 6 \\ 4 & 24 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}} = \frac{48 - 24}{0} = \frac{24}{0}$$



Division by zero is not allowed. *These equations are inconsistent.*

Example 3. Consider the system $\begin{cases} 2x + 3y = 6 \\ 4x + 6y = 12 \end{cases}$

$$x = \frac{\begin{vmatrix} 6 & 3 \\ 12 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}} = \frac{36 - 36}{12 - 12} = \frac{0}{0}; \quad y = \frac{\begin{vmatrix} 2 & 6 \\ 4 & 12 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}} = \frac{24 - 24}{0} = \frac{0}{0}$$

These two equations are *dependent.*

Rule. *If the denominator determinant is zero, the equations are either inconsistent or dependent. They are dependent if, besides, both numerators are zero; they are inconsistent if at least one numerator is not zero.*

EXERCISE 54. *(Optional)*

107. Solve by **determinants the following systems.** If they are not simultaneous, tell whether they are inconsistent or dependent.

1. $\begin{cases} a + b = 5 \\ a - b = 9 \end{cases}$

2. $\begin{cases} x + 2y = 12 \\ 3x - y = 1 \end{cases}$

3. $\begin{cases} 3x - 2y = -12 \\ 4x + 2y = -2 \end{cases}$

4. $\begin{cases} 2x + y = 4 \\ 6x + 3y = 12 \end{cases}$

5. $\begin{cases} 5x + 2y = 10 \\ x - y = 9 \end{cases}$

6. $\begin{cases} 3a - 2b = 6 \\ 6a - 4b = 18 \end{cases}$

7. $\begin{cases} 5x - 6y = 8 \\ 9x - 4y = -6 \end{cases}$

8. $\begin{cases} 8p + 5q = 5 \\ 3p - 2q = 29 \end{cases}$

9. $\begin{cases} 2x - 3y = -14 \\ 3x + 7y = 48 \end{cases}$

10. $\begin{cases} r - 6s = 2 \\ 2r - 12s = 10 \end{cases}$

108. Determinants are especially useful in solving sets of equations in three or more variables. *(Optional)*

To solve a set of three equations with three variables, determinants of three rows and three columns are used.

$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is a determinant of **the third** order. It is defined to have the value $a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$

This value may be obtained from the determinant as follows: *From the sum of the products of the downward to right diagonals subtract the products of the upward to right diagonals.*

Example.

$$\begin{vmatrix} 1 & 5 & 2 \\ 4 & 7 & 3 \\ 2 & -3 & 6 \end{vmatrix} = 1 \cdot 7 \cdot 6 + 5 \cdot 3 \cdot 2 + 2(-3) \cdot 4 - 2 \cdot 7 \cdot 2 - 4 \cdot 5 \cdot 6 - 1 \cdot 3(-3)$$

$$= 42 + 30 - 24 - 28 - 120 + 9, \text{ or } -91$$

$$= -91.$$

EXERCISE 55

Find the value of:

1. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 3 & 1 \end{vmatrix}$ **2.** $\begin{vmatrix} 2 & 4 & 6 \\ 3 & -2 & 3 \\ 1 & 5 & 4 \end{vmatrix}$ **3.** $\begin{vmatrix} 2 & 2 & 3 \\ -2 & -4 & -11 \\ 5 & -6 & 2 \end{vmatrix}$

4. Solve by determinants the equations:

$$\begin{cases} 3x + y - z = 14 \\ x + 3y - z = 16 \\ x + y - 3z = -10 \end{cases}$$

Solution. A rule similar to that of § 106 applies for linear equations with more than two unknowns. Namely:

$$x = \frac{\begin{vmatrix} 14 & 1 & -1 \\ 16 & 3 & -1 \\ -10 & 1 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & -3 \end{vmatrix}} = \frac{-126 + 10 - 16 - 30 + 48 + 14}{-27 - 1 - 1 + 3 + 3 + 3} = \frac{-100}{-20} = 5.$$

$$y = \frac{\begin{vmatrix} 3 & 14 & -1 \\ 1 & 16 & 1 \\ 1 & -10 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & -3 \end{vmatrix}} = \frac{-120}{-20} = 6. \qquad z = \frac{\begin{vmatrix} 3 & 1 & 14 \\ 1 & 3 & 16 \\ 1 & 1 & -10 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & -3 \end{vmatrix}} = \frac{-140}{-20} = 7.$$

Check. The solution checks when substituted in the three equations.

NOTE. The equations must be arranged first in the form $ax + by + cz = d$. Thus the equation $2x - 3z = 7$ would be written $2x + 0y - 3z = 7$.

Solve the following equations by determinants:

5. $\begin{cases} x + y - z = 24 \\ 4x + 3y - z = 61 \\ 6x - 5y - z = 11 \end{cases}$ **7.** $\begin{cases} 4x - 3y = 1 \\ 4y - 3z = -15 \\ 4z - 3x = 10 \end{cases}$

6. $\begin{cases} 5x - y + 4z = -5 \\ 3x + 5y + 6z = -20 \\ x + 3y - 8z = -27 \end{cases}$ **8.** $\begin{cases} 9x + 5z = -7 \\ 3x + 5y = 1 \\ 9y + 3z = 2 \end{cases}$

EXERCISE 56

1. If the numerator of a certain fraction be trebled, and the denominator be increased by 7, the value of the resulting fraction is $\frac{1}{4}$. If the denominator be doubled and the numerator be increased by 3, the value of the resulting fraction is $\frac{7}{10}$. Find the fraction.

2. The perimeter of a certain triangle is 100 inches. The sum of the first and second sides exceeds the third side by 12. The sum of the second and third sides equals four times the first side. What are the lengths of the sides of the triangle?

3. The sum of the reciprocals of certain three numbers is $-\frac{1}{6}$. The reciprocal of the first exceeds that of the second by $\frac{1}{6}$. The reciprocal of the first, diminished by twice that of the third, is $\frac{5}{2}$. What are the numbers?

4. If a rectangular lot were 6 feet longer and 5 feet wider than it is now, it would contain 839 square feet more; if it were 4 feet longer and 7 feet wider, it would contain 879 square feet more. Find its length and its width.

5. In a certain triangle, ABC, angle A exceeds angle B by 15°. The sum of angle A and twice angle C is 150°. Find the size of each angle.

6. A crew can row 15 miles downstream in 2 hours, and back the same distance in 3 hours. What is the rate of the crew in still water, and the rate of the current?

7. If A and B do a piece of work in 10 days, B and C in 20 days, and A and C in 12 days, how many days would it take each to do it alone?

8. The sum of the digits of a certain number of three digits is 17. The tens' digit exceeds the units' digit by 5. If the digits be reversed, the old number is 38 more than three times the new number. Find the number.

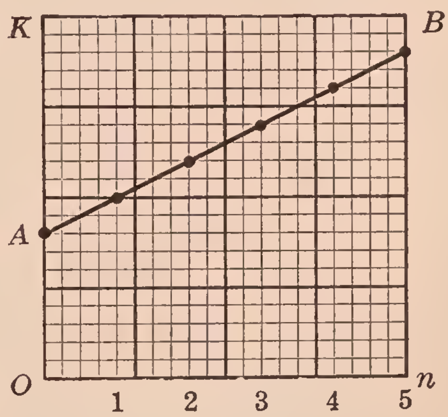
109. We are now able to obtain without any guessing the formula connecting two numbers when we have a table of related values of these numbers, — that is for certain cases. For other cases we must wait until later. This is one of the very practical uses of graphs and functional relationships in applied mathematics.

Example. Assume that an experiment involving the measurement of two numbers K and n has produced the following table of related values.

When $n =$	1,	3,	4,	5
then $K =$	10,	14,	16,	18.

What formula, if any, connects them?

Solution. 1. First draw the graph for these pairs of numbers.



2. The points appear to fall on a straight line. This suggests that the formula is of the form

$$K = an + b.$$

In this formula, a and b are unknown.

3. Substitute two sets of values from the table in this formula.

Thus

when $n = 1$, $K = 10$. $\therefore \quad a + b = 10.$

when $n = 3$, $K = 14$. $\therefore\ 3a + b = 14.$

4. Solving this system, $a = 2$ and $b = 8$.

Hence the formula appears to be $K = 2n + 8$.

5. Does it check? Try the pair of values $n = 5$ and $K = 18$. Does $18 = 2 \times 5 + 8$? Yes.

Try $n = 4$ and $K = 16$. Does $16 = 2 \times 4 + 8$? Yes.

Therefore the formula is $K = 2n + 8$.

NOTE. Understand, now, that the plan is to draw the graph for the table of related values. If the graph turns out to be a straight line, then we infer that one of the numbers is a first degree function of the other. If the graph is not a straight line, then the one variable is not a first degree function of the other, and we cannot solve the problem at the present time.

(Examples appear on page 96.)

EXERCISE 57

If possible, derive the formula for each of the following tables of related variables. Follow the plan taught on page 94.

1. (*a*) The parcel post rates to the third zone are:

weight	(w)	4 lb.	7 lb.	9 lb.	12 lb.
rate	(r)	14¢	20¢	24¢	30¢

Find the formula connecting w and r.

(*b*) From your formula give the rates for 1, 2, and 5 pounds.

(*c*) From your formula tell how r increases when w increases 1 lb.

2. (*a*) The rates for the sixth zone are:

weight	(w)	5 lb.	10 lb.	15 lb.	20 lb.
rate	(r)	42¢	82¢	1.22	1.62

(*b*) From your formula, find the rates for 1, 2, and 3 pounds.

(*c*) From your formula tell how r increases when w increases 1 lb.

3. (*a*) The rates for expressing a package not over 50 miles are:

weight	(w)	10 lb.	20 lb.	25 lb.
rate	(r)	43¢	51¢	55¢

(*b*) From your formula find the rates for 5 lb. and 15 lb.

Note. The rate is the same for 0 to 5 lb. inclusive; etc.

4. An object falling freely was found to pass over the following distances in the time given.

Time	1 second	3 seconds	4 seconds
Distance	16 ft.	80 ft.	112 ft.

Find the formula connecting the time t and the distance d.

5. After an experiment it was found that two numbers W and d are related as follows:

When $d =$	1	2	3	4
then $W =$	27	51	75	99

Find the formula connecting them, if possible.

EXERCISE 58. CHAPTER MASTERY-TEST

Solve the following systems of equations if possible and tell what kind of graphs they produce. If you cannot find a common solution, tell what kind of equations they are and what the graphs will be.

1. $\begin{cases} 3x + 5y = 19 \\ 4x - y = 10 \end{cases}$

2. $\begin{cases} 2x + y = 5 \\ 4x + 2y = 10 \end{cases}$

3. $\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} = 12 \\ \dfrac{3}{x} - \dfrac{1}{y} = 1 \end{cases}$

4. $\begin{cases} 6x - 2y = 15 \\ 3x - y = 4 \end{cases}$

5. $\begin{cases} 2x - y = 5a - b \\ 3x + 2y = 11a + 2b \end{cases}$

6. $\begin{cases} \dfrac{3x - y}{3} = \dfrac{4x - 2y - 8}{2} \\ \dfrac{2x - y}{3} = 5x + 3y \end{cases}$

Solve by determinants:

7. $\begin{cases} 4x - 3y = 1 \\ 3x - 4y = 6 \end{cases}$

8. $\begin{cases} a - 2b + c = 0 \\ a - b + 2c = -11 \\ 2a - b + c = -9 \end{cases}$

9. An equation like $-3x + 5y = 16$ expresses a ________ relation between ______ and ______. Solved for y, $y =$ ______. This last equation shows that an increase of 1 unit in x produces ________ of ______ units in y.

10. The functional relation $y = .5x + 7$ shows that a decrease of 1 unit in x produces a ________ of ______ units in y. The graph is a ________ line which rises ______ spaces for each space to the right; it crosses the y-axis at the point $y =$ ______ and the x-axis at the point $x =$ ______.

11. What formula connects w and s if:

when $s =$	5	8	10	13
then $w =$	3.5	3.8	4	4.3

12. If a two digit number be divided by the sum of its digits the quotient is 5 and the remainder is 8. If the digits be reversed, and the new number be divided by the sum of its digits, the quotient is 4 and the remainder is 6. What is the number?

EXERCISE 59

CUMULATIVE REVIEW

1. Factor: (*a*) $x^{2a} - 13x^a + 36$; (*c*) $12a^4 - 27b^4$;
(*b*) $6x^2 + xy - 35y^2$; (*d*) $x^{2c} - 2x^cy^d + y^{2d}$.

2. Write as a single fraction in its lowest terms:

$$\left(1 + \frac{5a+4}{a^2-3a-4}\right) \cdot \left(1 + \frac{a-16}{a^2-a}\right) \div \left(\frac{a^2+6a+8}{a^2+a}\right).$$

3. Find the numerical value of $\dfrac{6a^2 - a - 1}{3a+2}$ when $a = \frac{1}{3}$.

4. Solve the formula $\dfrac{a}{g} = \dfrac{h-L}{h+L}$ for L. Find the value of L, correct to the nearest tenth, when $a = 5.2$, $g = 32.16$, and $h = 7.5$.

5. Reduce to lowest terms: $\dfrac{\dfrac{2a-b}{a+3b} + \dfrac{2a+b}{a-3b}}{\dfrac{2a-b}{a-3b} - \dfrac{2a+b}{a+3b}}$. Check.

6. Solve the systems of equations:

(*a*) $\begin{cases} \dfrac{3}{x-1} + \dfrac{4}{y-1} = 0 \\ \dfrac{5}{2x-3} - \dfrac{7}{2y+13} = 0 \end{cases}$ (*b*) $\begin{cases} 12m - 4n + p = 3 \\ m - n - 2p = -1 \\ 5m - 2n = 0 \end{cases}$

7. Find two numbers whose sum is m and difference is n.

8. The sum of the three digits of a certain number is 13. If the number, decreased by 8, be divided by the sum of its units' and its tens' digits, the quotient is 25; if 99 be added to the number, the result has the same digits as the original number, but in reverse order. Find the number.

9. Solve the systems:

(*a*) $\begin{cases} x - ay = ab - 2a \\ bx + ay = -ab \end{cases}$ (*b*) $\begin{cases} mx - ny = 0 \\ mn(x-y) = m^2 - n^2 \end{cases}$

10. Solve the formulas $mg - T = mf$ and $T = nf$ for T and f.

VII. SQUARE ROOT AND QUADRATIC SURDS

110. To study the next chapter intelligently, you must know how to find the square roots of certain numbers. Possibly you remember how to do this from your previous course in mathematics. The following diagnostic test will help you discover what you do remember and what you must study with special care.

DIAGNOSTIC TEST

1. What is the square root of 25? Why?

2. Is there more than one square root of 25?

3. Can you give the positive square root of each of the following: 81? 144? 196? 169? 121? 225? 256?

4. What is $\sqrt{64\,a^2}$?

5. What is $(36\,x^2)^{\frac{1}{2}}$?

6. Find the square root of 2304.

7. Find the square root of 137 to two decimal places.

8. If $\sqrt{6} = 2.45$, how much is $\sqrt{150}$ to two decimal places.

9. If $\sqrt{10} = 3.162$, how much is $\sqrt{\frac{2}{5}}$, to two decimal places.

10. If $\sqrt{6} = 2.45$, how much is $\frac{1}{3} + \sqrt{\frac{1}{6}}$, to two decimal places.

NOTE. If you cannot do all these examples you need some of the Remedial Instruction and Practice which appears on pages 100 to 104. If you can do all these examples you can spend more time on the new material which appears on pages 105 to 113.

NOTE TO THE TEACHER. Only square roots appear in this chapter. Chapter XII can be studied now if the teacher wishes to teach cube root at this time.

Remedial Instruction and Practice

111. The square root of a number is one of two equal factors whose product is the given number.

Since $(+3a)(+3a) = +9a^2$, $+3a$ is a square root of $9a^2$.

Since $(-3a)(-3a) = +9a^2$, $-3a$ is a square root of $9a^2$.

Every number has two square roots. They are written together by means of the double sign, $\pm$, read *plus or minus.*

The positive square root is called the **principal square root. The** square root means the principal square root.

The principal square root of a number is indicated by the radical sign, $\sqrt{\ }$, or by the exponent $\frac{1}{2}$.

So $\sqrt{25} = +5$, and also $25^{\frac{1}{2}} = +5$.

The number under the radical sign is called the **radicand.**

112. Recognizing some perfect squares at sight.

(*a*) *Perfect square arithmetical numbers.* You should memorize, if you have not, the squares of the numbers 1 to 15, or 1 to 20. The more you know the better.

(*b*) *Perfect square monomials* have perfect square arithmetical coefficients and literal factors which have even exponents.

113. First principle employed in extracting roots.

The square root (or any other root) *of a product of two or more numbers is the product of the square roots of the numbers.*

Thus: $\sqrt{1746\,a^2} = \sqrt{4 \cdot 441\,a^2} = \sqrt{4} \cdot \sqrt{441} \cdot \sqrt{a^2} = 2 \cdot 21 \cdot a$, or $42\,a$.

EXERCISE 60. REMEDIAL PRACTICE

Give the principal square root of:

1. $49\,x^2$	5. $100\,x^2y^2$	9. $900\,a^2b^2$	13. $196\,m^2$
2. $64\,a^4$	6. $81\,x^2y^4$	10. $225\,m^2$	14. $1089\,x^6$
3. $121\,x^6$	7. $169\,a^6$	11. $2500\,x^2$	15. $1936\,a^2b^2$
4. $144\,r^2$	8. $400\,x^2$	12. $625\,a^4$	16. $2304\,m^4$

114. The square root of a perfect square polynomial can be found by a long division process. *(Optional)*

Example. Find the square root of $9x^2 + 25y^2 - 30xy$.

Solution. 1. Arrange the polynomial in descending powers of x.

2. $\sqrt{9x^2} = 3x$. Write $3x$ in the root.

3. Square $3x$, getting $9x^2$. Write it below $9x^2$.

4. Subtract, obtaining the first remainder.

5. Trial divisor, $2 \cdot 3x = 6x$.
$-30xy \div 6x = -5y$.
Write $-5y$ in the root and add $-5y$ to $6x$, forming the complete divisor.

6. Multiply the complete divisor by $-5y$.

7. Subtract. In this case, no remainder.

8. The *square roots* are $+(3x - 5y)$ and $-(3x - 5y)$.

$$
\begin{array}{r|l}
\text{Root} & 3x \qquad\qquad -5y \\
 & 9x^2 - 30xy + 25y^2 \\
 & 9x^2 \\ \hline
6x & -30xy + 25y^2 \\
-5y & \\ \hline
6x - 5y & -30xy + 25y^2 \\ \hline
\end{array}
$$

EXERCISE 61. *(Optional)*

Find the square root of:

1. $25x^2 + 40xy + 16y^2$
2. $9a^2 - 30ab + 25b^2$
3. $36c^2 - 60cd + 25d^2$
4. $49m^4 - 28m^2n + 4n^2$
5. $25r^2 - 70rs + 49s^2$
6. $4x^4 + 4x^3 + 5x^2 + 2x + 1$
7. $4x^4 - 4x^3 + 5x^2 - 2x + 1$
8. $9x^4 + 6x^3 + 13x^2 + 4x + 4$
9. $9a^4 + 12a^3 - 2a^2 - 4a + 1$
10. $y^4 - 6y^3 + 13y^2 - 12y + 4$
11. $4c^4 - 12c^3d + 17c^2d^2 - 12cd^3 + 4d^4$
12. $9m^4 - 24m^3n + 10m^2n^2 + 8mn^3 + n^4$
13. $1 - 2x - x^2 + 2x^3 + x^4$
14. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$
15. $4a^2 + 9b^2 + c^2 + 12ab - 4ac - 6bc$
16. $9a^2 - 24ab + 30ac + 16b^2 - 40bc + 25c^2$
17. $4x^2 - 12xy - 16xz + 9y^2 + 24yz + 16z^2$

NOTE. Not required by C.E.E.B.

115. Finding the square root of an arithmetical number.

Example 1. Find the principal square root of 4624.

Solution. 1. Since 4624 is more than 3600, $\sqrt{4624}$ is more than 60; and since 4624 is less than 4900, 4624 is less than 70.

		a		*b*
		60 + 8		6 8
		46 24		46 24
1. $60^2 = 3600$. Subtract from 4624.		36 00		36
2. $2 \times 60 = 120$; $1024 \div 120 = 8^+$	120	10 24	120	10 24
3. Add 8 to 120 and to 60.	8		8	
4. Multiply 128 by 8. Subtract.	128	10 24	128	10 24

NOTE 1. Use only the form *b* when writing the solution.

NOTE 2. 4624 was separated into groups of two figures each because the square of any number contains as many groups of two figures each as there are digits in the given number — except that the left-most group may have only one digit in it.

Thus: $3^2 = 9$; $8^2 = 64$; $12^2 = 144$; $95^2 = 9025$.

Similarly 34,038 will be written 3 40 38. This shows that its square root has three figures to the left of the decimal point.

A number written in decimal form is divided in the same manner, counting in both directions from the decimal point. Thus, 34256.895 becomes 3 42 56.89 50. The square root of this number has three figures to the left of the decimal point.

Example 2. Find the principal square root of 5207.0656.

Solution. 1. The largest square number in 52 is 49. Write $\sqrt{49}$ or 7 in the root. Subtract. Annex 07.

2. Trial divisor is 2×7. Annex 0. $307 \div 140 = 2^+$. Place 2 in the root. Add 2 to 140. Multiply 142 by 2. Subtract. Annex 06.

3. Trial divisor is 2×72. Annex 0. $2306 \div 1440 = 1^+$. Place 1 in the root. Add 1 to 1440. Multiply 1441 by 1. Subtract. Annex 56.

4. Trial divisor is 2×721. Annex 0. $86556 \div 14420 = 6^+$. Place 6 in the root. Add 6 to 14420. Multiply 14426 by 6. No remainder.

	7 2.1 6
	52 07.06 56
	49
140	3 07
2	
142	2 84
1440	23 06
1	
1441	14 41
14420	8 65 56
6	
14426	8 65 56

NOTE. The root, *correct to tenths*, is 72.2.

Rule. To find the principal square root of a number:

1. *Separate the number into groups (periods) of two figures each, starting at the decimal point, and forming the groups each way from the decimal point.*

2. *Find the largest square number not more than the left-most period. Write its square root as the first figure of the square root. Subtract the square number itself from the first period.*

3. *To the remainder, annex (bring down) the next period.*

4. *Form the trial divisor by doubling the part of the root already found, and annexing a zero.*

5. *Divide the remainder formed in Step 3 by the trial divisor. Annex the quotient to the part of the root already found, and add it to the trial divisor to form the complete divisor.*

6. *Multiply the complete divisor by the new figure of the root. Subtract the result from the remainder formed in Step 3.*

7. *If other periods remain, repeat Steps 4, 5, and 6 until there is not a remainder, or until the desired number of decimal places for the root have been obtained.*

NOTE. In Step 6, if the product is greater than the remainder, the last figure obtained for the root is too large.

116. Only the approximate square root can be found in most cases. For example, $\sqrt{20}$ does not have an exact square root. In such cases, we carry out the square root to as many decimal places as we please, annexing pairs of zeros as they are needed.

EXERCISE 62. REMEDIAL PRACTICE

Find the principal square root, correct to hundredths:

1. 4489	**6.** 16,129	**11.** 13	**16.** 4.7524
2. 8836	**7.** 27,556	**12.** 91	**17.** 806.56
3. 5625	**8.** 11,664	**13.** 165	**18.** 46.3761
4. 9604	**9.** 95,481	**14.** 23	**19.** 723.61
5. 6889	**10.** 214,369	**15.** 463	**20.** 568.25

117. A table of square roots can be used to advantage. A short table appears on page 253. Often the square root of a number not in the table can be found by using a number in the table.

Thus: $\sqrt{132} = \sqrt{4 \cdot 33} = \sqrt{4} \cdot \sqrt{33} = 2 \times 5.7446$, or 11.4892.
Correct to hundredths, $\sqrt{132} = 11.49$.

You will find it a big help if you memorize the three place values of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{6}$. (See page 253.)

118. The **square root of a fraction equals** *the square root of its numerator divided by the square root of its denominator.* However, always change the fraction to an equal fraction whose denominator is the smallest perfect square into which the given denominator can be changed easily.

Example. $\sqrt{\dfrac{3}{2}} = \sqrt{\dfrac{6}{4}} = \dfrac{\sqrt{6}}{2} = \dfrac{2.449}{2} = 1.224 = 1.22$

Find the following correct to hundredths:

1. $\sqrt{20}$	3. $\sqrt{108}$	5. $\sqrt{125}$	7. $\sqrt{175}$	9. $\sqrt{180}$
2. $\sqrt{27}$	4. $\sqrt{112}$	6. $\sqrt{200}$	8. $\sqrt{147}$	10. $\sqrt{192}$
11. $\sqrt{\frac{3}{4}}$	13. $\sqrt{\frac{4}{3}}$	15. $\sqrt{\frac{3}{8}}$	17. $\sqrt{\frac{5}{18}}$	19. $\sqrt{\frac{9}{20}}$
12. $\sqrt{\frac{2}{5}}$	14. $\sqrt{\frac{5}{6}}$	16. $\sqrt{\frac{7}{12}}$	18. $\sqrt{\frac{11}{5}}$	20. $\sqrt{\frac{2}{27}}$

21. Find the simplest radical form and also the decimal value correct to hundredths of $\frac{1}{3} - \sqrt{\frac{1}{2}}$.

Solution. $\dfrac{1}{3} - \sqrt{\dfrac{1}{2}} = \dfrac{1}{3} - \sqrt{\dfrac{2}{4}} = \dfrac{1}{3} - \dfrac{\sqrt{2}}{2} = \dfrac{2 - 3\sqrt{2}}{6}$

Since $\sqrt{2} = 1.414$,

$$\frac{2 - 3\sqrt{2}}{6} = \frac{2 - 4.242}{6} = \frac{-2.242}{6} = -.373 = -.37.$$

22. $\frac{1}{2} - \sqrt{\frac{3}{2}}$	26. $-\frac{1}{3} + \sqrt{\frac{1}{6}}$	30. $-\frac{1}{5} + \sqrt{\frac{2}{5}}$
23. $\frac{1}{3} + \sqrt{\frac{1}{2}}$	27. $-\frac{2}{3} - \sqrt{\frac{1}{3}}$	31. $+\frac{3}{8} - \sqrt{\frac{5}{8}}$
24. $\frac{1}{4} - \sqrt{\frac{1}{2}}$	28. $\frac{3}{4} - \sqrt{\frac{3}{8}}$	32. $-\frac{5}{6} + \sqrt{\frac{7}{18}}$
25. $\frac{3}{4} + \sqrt{\frac{1}{2}}$	29. $-\frac{1}{2} + \sqrt{\frac{5}{8}}$	33. $+\frac{3}{10} - \sqrt{\frac{7}{10}}$

Quadratic Surds

119. A **quadratic surd** is the indicated square root of a number which is not a perfect square; as $\sqrt{3}$.

120. A **surd expression** is an expression which involves one or more surds. In this chapter, only simple quadratic surd expressions will be considered.

121. A **surd quadratic is in its simplest form** when the number under the radical sign is not a fraction and does not have any perfect square factors.

122. To simplify a surd expression, first simplify the surds and then perform the indicated operations.

Example 1. Simplify $\sqrt{20} + \sqrt{45}$.

Solution. 1. $\sqrt{20} + \sqrt{45} = \sqrt{4 \cdot 5} + \sqrt{9 \cdot 5}$

2. $\qquad = 2\sqrt{5} + 3\sqrt{5}$, or $5\sqrt{5}$.

This is the result in simplest radical form.

3. $\therefore \sqrt{20} + \sqrt{45} = 5\sqrt{5} = 5 \cdot 2.236$, or 11.180.

Note. The surds $2\sqrt{5}$ and $3\sqrt{5}$ are called *similar surds,* because their surd factors are the same. *Only similar surds can be combined.*

Example 2. Simplify the surd $\sqrt{\dfrac{c^3d}{a+b}}$.

Solution. $\sqrt{\dfrac{c^3d}{a+b}} = \sqrt{\dfrac{c^3d(a+b)}{(a+b)^2}}$, or $\dfrac{c}{a+b}\sqrt{cd(a+b)}$.

Example 3. Find the simplest radical form and also the decimal value correct to hundredths of $\sqrt{\frac{9}{8}} + \sqrt{\frac{1}{2}}$.

Solution. 1. $\sqrt{\dfrac{9}{8}} + \sqrt{\dfrac{1}{2}} = \sqrt{\dfrac{9 \cdot 2}{16}} + \sqrt{\dfrac{2}{4}}$

2. $\qquad = \dfrac{3\sqrt{2}}{4} + \dfrac{\sqrt{2}}{2}$

3. $\qquad = \dfrac{3\sqrt{2} + 2\sqrt{2}}{4}$, or $\dfrac{5\sqrt{2}}{4}$.

4. $\dfrac{5\sqrt{2}}{4} = \dfrac{5 \times 1.414}{4} = \dfrac{7.07}{4} = 1.767$, or 1.77.

EXERCISE 64

Find the simplest radical form, and also the decimal value of the following expressions, correct to hundredths:

1. $\sqrt{12}+\sqrt{75}$
2. $\sqrt{98}-\sqrt{18}$
3. $\sqrt{80}-\sqrt{20}$
4. $\sqrt{45}+\sqrt{20}$
5. $2\sqrt{6}-\sqrt{54}$
6. $3\sqrt{27}-\sqrt{48}$
7. $2\sqrt{28}+\sqrt{63}$
8. $2\sqrt{20}-\sqrt{45}$
9. $+\frac{2}{3}-\sqrt{\frac{1}{3}}$
10. $-\frac{1}{2}-\sqrt{\frac{3}{2}}$
11. $+\frac{1}{3}-\sqrt{\frac{1}{6}}$
12. $+\frac{1}{2}-\sqrt{\frac{5}{8}}$
13. $-\frac{1}{3}+\sqrt{\frac{5}{6}}$
14. $+\frac{5}{2}-\sqrt{\frac{11}{12}}$
15. $+\frac{3}{5}+\sqrt{\frac{6}{5}}$
16. $-\frac{3}{4}-\sqrt{\frac{7}{8}}$
17. $-\frac{5}{3}+\sqrt{\frac{7}{6}}$
18. $+\frac{5}{2}-\sqrt{\frac{3}{10}}$
19. $-\frac{2}{5}-\sqrt{\frac{2}{15}}$
20. $+\frac{7}{4}-\sqrt{\frac{15}{8}}$
21. $\sqrt{\frac{2}{5}}+\sqrt{\frac{1}{10}}$
22. $\sqrt{\frac{2}{3}}-\sqrt{\frac{1}{27}}$
23. $\sqrt{\frac{3}{8}}-\sqrt{24}$
24. $\sqrt{\frac{3}{4}}-\sqrt{\frac{4}{3}}$
25. $2\sqrt{20}-\sqrt{45}+\sqrt{5}$
26. $\sqrt{50}-\sqrt{98}+\sqrt{72}$
27. $\sqrt{\frac{2}{9}}+\sqrt{\frac{1}{8}}-\sqrt{\frac{1}{2}}$
28. $\sqrt{\frac{4}{3}}+\sqrt{\frac{1}{27}}-\sqrt{\frac{1}{12}}$

Find the simplest radical form:

29. $\sqrt{9\,a}+\sqrt{16\,a}-\sqrt{25\,a}$
30. $\sqrt{9\,x^2yz}-\sqrt{x^2yz}$
31. $\sqrt{ax^2}-\sqrt{9\,ax^2}+\sqrt{4\,ax^2}$
32. $2\sqrt{25\,y}-\sqrt{36\,y}+5\sqrt{9\,y}$
33. $\sqrt{18\,a}-\sqrt{\frac{1}{2}\,a}+3\sqrt{50\,a}$
34. $a\sqrt{98}-\sqrt{75\,a^2}+3\,a\sqrt{32}$
35. $\dfrac{-2+\sqrt{8\,a^2-4}}{2}$
36. $\dfrac{+4\,c-\sqrt{16\,c^2-4\,d^2}}{4}$
37. $\dfrac{ab-\sqrt{ab^2-b^3}}{2\,b}$
38. $\dfrac{-6\,c-\sqrt{mc^2+nc^2}}{2\,c}$
39. $\dfrac{+at-\sqrt{at^2-t^3}}{at}$
40. $\dfrac{-2\,b-\sqrt{4\,b^2-4\,ac}}{2\,a}$
41. $\sqrt{\dfrac{a}{b^2}}$
42. $\sqrt{\dfrac{c}{d}}$
43. $\sqrt{\dfrac{m^2}{n}}$
44. $\sqrt{\dfrac{rs}{2\,t}}$
45. $\sqrt{\dfrac{v^2t}{g}}$
46. $\sqrt{\dfrac{\pi}{2\,r}}$
47. $\sqrt{\dfrac{a}{\pi}}$
48. $\sqrt{\dfrac{s}{4\,\pi}}$
49. $\sqrt{\dfrac{a-b}{3}}$
50. $\sqrt{\dfrac{x+y}{z}}$
51. $\sqrt{\dfrac{m^3}{x-y}}$
52. $\sqrt{\dfrac{a-b}{a+b}}$

123. Multiplication of simple quadratic surds.

Since $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, then also $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$.

Thus, $\sqrt{2} \cdot \sqrt{3} = \sqrt{2 \cdot 3}$, or $\sqrt{6}$.

Example. Find the simplest radical form and also the decimal value of $2\sqrt{3} \times 5\sqrt{2}$ correct to hundredths.

Solution. 1. $2\sqrt{3} \times 5\sqrt{2} = 10\sqrt{3 \cdot 2} = 10\sqrt{6}$.
2. $10\sqrt{6} = 10 \times 2.449 = 24.49$.

NOTE. Keep the product under the radical sign in factored form. Thus, $\sqrt{18} \cdot \sqrt{12} = \sqrt{18 \cdot 12} = \sqrt{9 \cdot 2 \cdot 4 \cdot 3} = 3 \cdot 2\sqrt{6}$, or $6\sqrt{6}$.

EXERCISE 65

Find the simplest radical form and also the decimal value of the following, to hundredths:

1. $\sqrt{2} \cdot \sqrt{8}$
2. $\sqrt{3} \cdot \sqrt{6}$
3. $\sqrt{5} \cdot \sqrt{10}$
4. $\sqrt{6} \cdot \sqrt{10}$
5. $\sqrt{7} \cdot \sqrt{14}$
6. $\sqrt{11} \cdot \sqrt{33}$
7. $3\sqrt{2} \cdot 4\sqrt{5}$
8. $2\sqrt{6} \cdot 5\sqrt{2}$
9. $5\sqrt{30} \cdot 3\sqrt{10}$
10. $(2\sqrt{3})^2$
11. $(3\sqrt{2})^2$
12. $(\frac{1}{3}\sqrt{3})^2$

13. Multiply $3 + 5\sqrt{2}$ by $3 + 5\sqrt{2}$.

Solution.

$$\begin{array}{l} 3 + 5\sqrt{2} \\ \underline{3 + 5\sqrt{2}} \\ 9 + 15\sqrt{2} \\ \underline{\quad\quad 15\sqrt{2} + 25\sqrt{4}} \\ 9 + 30\sqrt{2} + 25 \cdot 2 \text{ or } 59 + 30\sqrt{2} \end{array}$$

Find:

14. $(5 - \sqrt{3})(5 + \sqrt{3})$
15. $(5 + \sqrt{6})(5 - \sqrt{6})$
16. $(3 - \sqrt{2})(3 + \sqrt{2})$
17. $(9 - 2\sqrt{5})(9 + 2\sqrt{5})$
18. $(-7 + \sqrt{13})(-7 + \sqrt{13})$
19. $(10 - 3\sqrt{6})(10 + 3\sqrt{6})$
20. $(3 - 2\sqrt{2})(3 + 2\sqrt{2})$
21. $(5 + 3\sqrt{7})(5 - 3\sqrt{7})$
22. $(\sqrt{x - 5})^2$
23. $(\sqrt{2x + 3})^2$
24. $(\sqrt{-5 + x^2})^2$
25. $(2 - \sqrt{3})^2$
26. $(a - \sqrt{b})^2$
27. $(m + \sqrt{r})^2$
28. $(\sqrt{b^2 - 4ac})^2$
29. $(a - \sqrt{bc})(a + \sqrt{bc})$

124. Division of simple quadratic surds.

Rule. The quotient of the square roots of two numbers *equals the square root of the quotient of the numbers.*

Example 1. $6\sqrt{28} \div 2\sqrt{7} = \frac{6}{2}\sqrt{\frac{28}{7}} = 3\sqrt{4} = 3 \cdot 2$, or 6.

Example 2. $\sqrt{5} \div \sqrt{15} = \sqrt{\frac{5}{15}} = \sqrt{\frac{1}{3}} = \sqrt{\frac{3}{9}}$, or $\frac{1}{3}\sqrt{3}$.

EXERCISE 66

Find the following quotients.

1. $\sqrt{14} \div \sqrt{7}$
2. $\sqrt{50} \div \sqrt{10}$
3. $\sqrt{72} \div \sqrt{2}$
4. $\sqrt{45} \div \sqrt{5}$
5. $\sqrt{108} \div \sqrt{3}$
6. $5\sqrt{21} \div \sqrt{3}$
7. $4\sqrt{75} \div 2\sqrt{3}$
8. $6\sqrt{68} \div \sqrt{17}$
9. $2\sqrt{40} \div \sqrt{8}$
10. $3\sqrt{192} \div \sqrt{3}$
11. $\sqrt{27} \div 2\sqrt{3}$
12. $\sqrt{32} \div 3\sqrt{2}$
13. $\sqrt{6} \div \sqrt{24}$
14. $\sqrt{7} \div \sqrt{35}$
15. $\sqrt{45} \div 3\sqrt{5}$
16. $2\sqrt{99} \div 3\sqrt{11}$
17. $5\sqrt{8} \div \sqrt{6}$
18. $4\sqrt{3} \div \sqrt{5}$
19. $3\sqrt{2} \div \sqrt{12}$
20. $11\sqrt{5} \div \sqrt{15}$
21. $8\sqrt{6} \div 4\sqrt{18}$

125. Making a monomial divisor a rational number.

Example. $\dfrac{20}{\sqrt{12}} = \dfrac{20 \cdot \sqrt{3}}{\sqrt{12} \cdot \sqrt{3}} = \dfrac{20\sqrt{3}}{\sqrt{36}} = \dfrac{20\sqrt{3}}{6} = \dfrac{10\sqrt{3}}{3}.$

The numerator and denominator are both multiplied by a surd which makes the new denominator the square root of the smallest possible perfect square.

EXERCISE 67

1. $\dfrac{2}{\sqrt{3}}$
2. $\dfrac{4}{\sqrt{5}}$
3. $\dfrac{6}{\sqrt{2}}$
4. $\dfrac{12}{\sqrt{6}}$
5. $\dfrac{15}{\sqrt{3}}$
6. $\dfrac{8}{\sqrt{12}}$
7. $\dfrac{9}{\sqrt{18}}$
8. $\dfrac{20}{\sqrt{10}}$
9. $\dfrac{19}{\sqrt{19}}$
10. $\dfrac{999}{\sqrt{999}}$
11. $\dfrac{7}{\sqrt{14}}$
12. $\dfrac{25}{\sqrt{15}}$
13. $\dfrac{6}{\sqrt{27}}$
14. $\dfrac{14}{\sqrt{28}}$
15. $\dfrac{25}{\sqrt{50}}$
16. $\dfrac{33}{\sqrt{11}}$

126. Rationalizing a simple binomial divisor.

Example. Rationalize the denominator of $\frac{3\sqrt{2}+1}{2\sqrt{2}-1}$.

Solution. 1. $\frac{3\sqrt{2}+1}{2\sqrt{2}-1} = \frac{(3\sqrt{2}+1)(2\sqrt{2}+1)}{(2\sqrt{2}-1)(2\sqrt{2}+1)} = \frac{12+5\sqrt{2}+1}{8-1}$.

2. $\therefore \frac{3\sqrt{2}+1}{2\sqrt{2}-1} = \frac{13+5\sqrt{2}}{7} = \frac{13+5(1.414)}{7} = 2.9.$

NOTE. The surd expressions $2\sqrt{2}+1$ and $2\sqrt{2}-1$ are called conjugate surds.

Rule. To rationalize a binomial surd denominator, *multiply both numerator and denominator by the conjugate of the denominator.*

EXERCISE 68

1. $\frac{6}{3+\sqrt{5}}$

2. $\frac{5}{\sqrt{5}-2}$

3. $\frac{4}{3-\sqrt{7}}$

4. $\frac{6}{\sqrt{3}-2}$

5. $\frac{7}{2\sqrt{2}-1}$

6. $\frac{11}{2\sqrt{3}+2}$

7. $\frac{9}{5-2\sqrt{3}}$

8. $\frac{10}{6-3\sqrt{2}}$

9. $\frac{\sqrt{2}+1}{\sqrt{2}-1}$

10. $\frac{2-\sqrt{3}}{2+\sqrt{3}}$

11. $\frac{3-\sqrt{2}}{3+\sqrt{2}}$

12. $\frac{4+\sqrt{5}}{4-\sqrt{5}}$

13. $\frac{6+\sqrt{2}}{\sqrt{2}-1}$

14. $\frac{8-\sqrt{3}}{2-\sqrt{3}}$

15. $\frac{10+\sqrt{5}}{3+\sqrt{5}}$

16. $\frac{\sqrt{6}-5}{\sqrt{6}-2}$

17. $\frac{\sqrt{8}+5}{\sqrt{8}-5}$

18. $\frac{\sqrt{7}-2}{\sqrt{7}+2}$

19. $\frac{2\sqrt{2}-1}{2\sqrt{2}+1}$

20. $\frac{2\sqrt{3}-1}{\sqrt{3}-1}$

21. $\frac{2-3\sqrt{2}}{3-\sqrt{2}}$

22. $\frac{2\sqrt{5}-1}{\sqrt{5}+2}$

23. $\frac{\sqrt{6}-2}{2\sqrt{6}+3}$

24. $\frac{3\sqrt{6}-5}{2\sqrt{6}-1}$

Radical Equations

127. A **radical equation** is an equation in which the variable appears under a radical sign or with a fractional exponent.

The radical sign or the fractional exponent always indicates the principal root in a radical equation.

Example 1. Solve the equation $x - 1 - \sqrt{x^2 - 5} = 0$.

Solution. 1. $x - 1 - \sqrt{x^2 - 5} = 0.$

2. $\therefore x - 1 = \sqrt{x^2 - 5}.$

3. Squaring both members, $x^2 - 2x + 1 = x^2 - 5.$

4. $\therefore -2x = -6$, or $x = 3$.

Check. Does $3 - 1 - \sqrt{9 - 5} = 0$? Does $2 - \sqrt{4} = 0$? Yes.

Note. When a single radical occurs in an equation, transpose the radical to one side and all other terms to the other side. Then, *if the radical is a square root, square both members of the equation; if it is a cube root, cube both members;* etc.

Example 2. Solve the equation $x - 1 - \sqrt{x^2 - 5} = 0$.

Solution. 1. $x - 1 + \sqrt{x^2 - 5} = 0.$

2. $\therefore +\sqrt{x^2 - 5} = 1 - x.$

3. Squaring, $x^2 - 5 = 1 - 2x + x^2.$

4. $\therefore 2x = 6$, or $x = 3$.

Check. Does $3 - 1 + \sqrt{9 - 5} = 0$? Does $2 + \sqrt{4} = 0$? No.

Therefore 3 *is not a root of the given equation.*

What is the explanation of the solution $x = 3$? If the original equation is compared with the equation of Example 1, it is noticed that the only difference is in the sign of the radical; also that in Step 3, after squaring both members in both examples, the resulting equations are the same. In each example, *if the equation of Step* 1 *has a root,* that number is a root of the equation of Step 3; but, since the equation of Step 3 is the same in each solution, it cannot be asserted in advance whether its root or roots are roots of the equation of Example 1 or of Example 2. When finally the solution $x = 3$ is obtained, the question arises, is 3 a root of the equation in Example 1 or in Example 2? The root $x = 3$ satisfies the equation of Example 1; it does not satisfy the equation of Example 2. In Example 2, the *extraneous root* 3 *is introduced by the method of solution.* In Example 2, Step 2 really is $\sqrt{x^2 - 5} = -(x - 1)$. When squaring, the minus sign in front of $(x - 1)$ is lost.

EXERCISE 69

Solve and check the following equations:

1. $\sqrt{7a+2} - 4 = 0$
2. $4 - 3\sqrt{y} = -4$
3. $\sqrt{7x-3} + 3 = 2x$
4. $\sqrt{x-1} = 2$
5. $\sqrt{x+1} = -3$
6. $\sqrt{2x-5} = +3$
7. $-\sqrt{x+3} = -3$
8. $-\sqrt{x+4} = 4$
9. $\sqrt{1-x} = 2$
10. $-\sqrt{-1-2x} = -3$
11. $\sqrt{13-4x} = 2x+11$
12. $\sqrt{5x^2-6x+9} = 10$
13. $\sqrt{9x^2-5x+1} = 3x$
14. $\sqrt{3x+1} = \sqrt{11+x}$
15. $\sqrt{10-3x} = 4-x$
16. $\sqrt{4x+5} = 3x+4$
17. $\sqrt{1-5y} + y = 1$
18. $\sqrt{8x+5} - 1 = 4x$
19. $\sqrt{15x+11} = 3(x+1)$
20. $\sqrt{12x+7} = 12x-5$
21. Solve the equation $t = \pi\sqrt{\frac{l}{g}}$
    (*a*) for l (*b*) for g
22. Solve the equation $V = \sqrt{2gs}$
    (*a*) for g (*b*) for s
23. Solve for s: $\frac{3r}{2} = \sqrt{s^2 - \left(\frac{s}{2}\right)^2}$
24. Solve for h: $L = \pi r\sqrt{r^2+h^2}$

Solve for x:

Suggestion. First clear of fractions.

25. $\sqrt{x-12ab} = \frac{9a^2-b^2}{\sqrt{x}}$
26. $\sqrt{a+x} - \sqrt{2x} = \frac{2a}{\sqrt{a+x}}$
27. $\sqrt{x+3} + \frac{x+3}{\sqrt{x}} = \frac{24}{\sqrt{x}}$
28. $\sqrt{x} + \sqrt{x-9} = \frac{36}{\sqrt{x-9}}$
29. $\frac{10x}{\sqrt{10x-9}} + \sqrt{10x+2} = \frac{2}{\sqrt{10x-9}}$
30. $\frac{\sqrt{t+1}}{\sqrt{t+3}} = \frac{\sqrt{t+3}}{\sqrt{t+6}}$

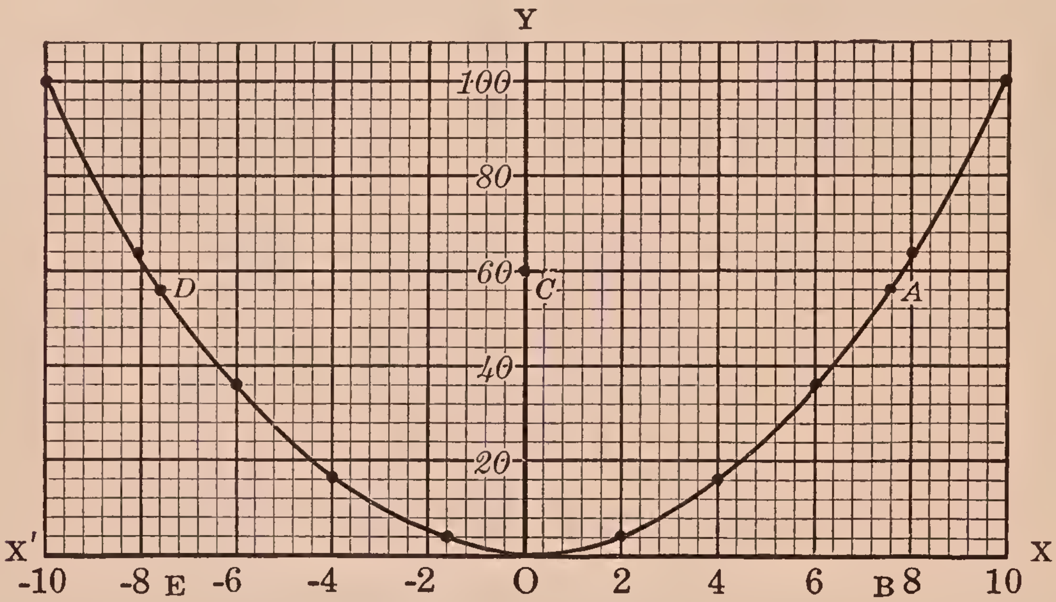
NOTE. Optional examples appear on p. 245.

128. (*Optional.*) **There is a functional relation between a number and its square root,** because for every number there is a square root. In fact there are two square roots. This is something new in a functional relation. It is interesting to study it graphically.

Let $$y = x^2.$$

This means that $$x = \pm \sqrt{y}.$$

When $x =$	0	± 2	± 4	± 6	± 8	± 10
then $y =$	0	$+ 4$	$+ 16$	$+ 36$	$+ 64$	$+ 100$



Notice: 1. For every value of x there is one value of y.

Thus when $x = 7$, $y = 49$.

2. But for every value of y there are two values of x.

Thus let $y = 56$. From point C, go to the right to point A on the graph, and then down to the x-axis at the point whose abscissa is about $+ 7.5$. This is the positive square root of 56.

Similarly go from point C to the left to point D on the graph, and then down to point E whose abscissa is about $- 7.5$. This is the negative square root of 56.

NOTE. For convenience the y-unit is one tenth as large as the x-unit. This flattens the curve considerably.

EXERCISE 70. CUMULATIVE REVIEW

1. Solve for x: $\frac{x+1}{x+3} = \frac{x+4}{x+2}$. Check.

2. Solve for x: $3 - \frac{5-2x}{5} = 4 - \frac{4-7x}{10} + \frac{x+2}{2}$.

3. Solve for x: $3cx - 5a + b - 2c = 6b - (a + 3bx + 2c)$.

4. The sum of the digits of a number of two digits is 9. If the digits be reversed, the new number is only three eighths as large as the old. Find the number.

5. A crew can row 4 miles against the current in forty minutes and the same distance back again in twenty minutes. What is the rate of the current and of the crew in still water?

6. One hundred pounds of an alloy of silver and copper contains 2 parts of silver to 3 of copper. How much copper must be melted with the alloy so that the resulting alloy will contain 3 parts of silver to 5 of copper?

7. Solve the system $\begin{cases} 6x - 5y = 25 \\ 4x - 3y = 19 \end{cases}$

8. What kind of equations are:

(a) $\begin{cases} 2y - 3x = 0 \\ 3x - 2y = 5 \end{cases}$ (b) $\begin{cases} 3x - 2y = 5 \\ 6x - 4y = 10 \end{cases}$

9. Solve the system $\begin{cases} ax + by = c \\ px - qy = 0 \end{cases}$

10. (a) Solve the formula $D = \frac{W}{W - w}$ for w.

(b) If W is constant, and D increases, how does w change?

11. Find, correct to hundredths $\frac{5}{9} - \sqrt{\frac{5}{6}}$

12. Solve the system: $\begin{cases} 3x - 2y + 4z = 13 \\ 2x + 5y - 3z = -9 \\ 6x + 3y + 2z = 7 \end{cases}$

VIII. QUADRATIC FUNCTIONS AND EQUATIONS

One Variable

129. An interesting formula from Physics. You know the saying that "anything that goes up must come down." The formula $S = at - 16t^2$ will help you find approximately how high up "it" will go, how long it will take to get there and when it will "come down," provided you know the rate at which it started "up." In this formula:

a = the rate in feet per second at which the object is started on its upward journey;

t = the number of seconds it travels;

S = the distance it goes in t seconds.

Let us take the case of a ball which is thrown upward at the rate of 100 feet per second.

Then $$S = 100t - 16t^2.$$

Evidently the distance S depends on the time t.

When $t = 0$, $S = 0$, since the ball is still on the ground.

When t =	0	1	2	3	4	5	6	7
then S =	0	84	136	156	144	100	24	– 84

The graph on page 115 is the result of locating points whose coördinates are the corresponding pairs of numbers in this table.

Remember that, on this graph, S represents the distance up from the ground, and that t is the number of seconds it takes the ball to travel this distance S.

Thus, at the end of 1 second, the ball is up 84 feet. At some time between 3 and 4 seconds, the ball reaches its highest position, which is a little more than 156 feet.

Question 1. When does the ball reach the height of 100 feet? That is, for what value of t is $100\,t - 16\,t^2 = 100$?

Answer. At A, $S = 100$ ft. and $t =$ about $1\frac{1}{4}$ sec.

At B, $S = 100$, and $t = 5$.

So $1\frac{1}{4}$ and 5 both satisfy the equation $100\,t - 16\,t^2 = 100$.

Question 2. When is the ball up 24 feet?

That is, for what value of t is $100\,t - 16\,t^2 = 24$?

Answer. At C, $S = 24$ and $t = \frac{1}{4}$; at D, $t = 6$.

Both of these values satisfy the equation $100\,t - 16\,t^2 = 24$.

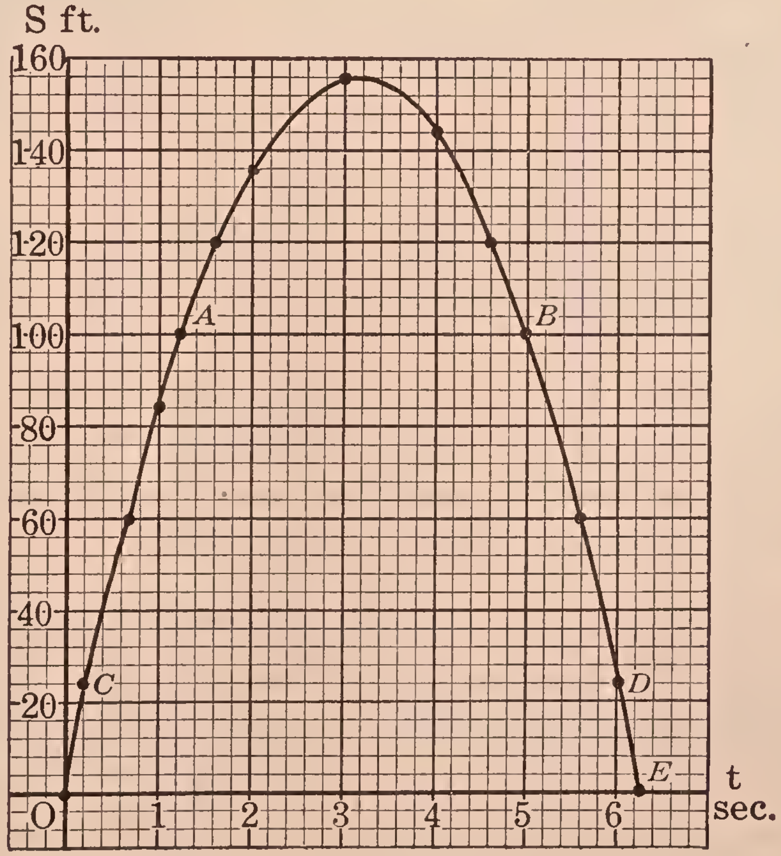
Question 3. When is the ball on the ground?

That is for what value of t is $100\,t - 16\,t^2 = 0$?

Answer. At E, $S = 0$, and $t =$ about $6\frac{1}{4}$.

Also when $t = 0$, $S = 0$, or $100\,t - 16\,t^2 = 0$.

There are two values of t which satisfy each of the equations.



130. $100\,t - 16\,t^2$ is a **second degree function** of t.

The equations $100\,t - 16\,t^2 = 100$; $100\,t - 16\,t^2 = 24$; and $100\,t - 16\,t^2 = 0$ are **second degree** or **quadratic equations.**

Observe that the variable t does not appear in a denominator or under a radical sign and that its highest exponent is 2.

Since there is a term containing the first power of t, $100\,t - 16\,t^2 = 100$ is a **complete quadratic equation.**

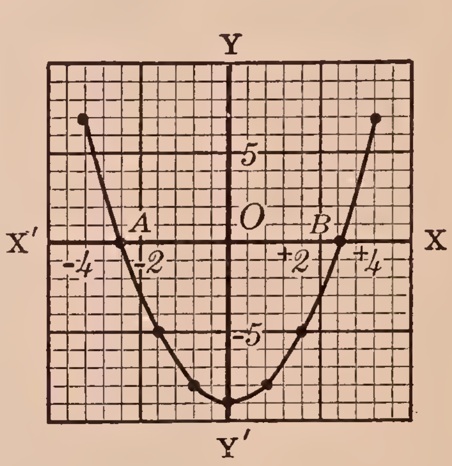
Similarly $3\,x^2 + x - 5 = 0$ is a *complete quadratic equation.*

But $3\,x^2 - 5 = 0$ is an **incomplete quadratic equation.**

131. Solving the incomplete quadratic equation $x^2 - 9 = 0$.

(*a*) *Graphical solution.* 1. Let $y = x^2 - 9$

When x =	−4	−2	−1	0	1	+2	+4
Then y =	+7	−5	−8	−9	−8	−5	+7



2. At A: $y = 0$; $x = -3$.
Does $(-3)^2 - 9 = 0$? Yes.
At B: $y = 0$; $x = +3$.
Does $3^2 - 9 = 0$? Yes.

The equation $x^2 - 9 = 0$ has the two roots $+3$ and -3.

(*b*) *Algebraic Solution.* 1. $x^2 - 9 = 0$

2. A_9: $x^2 = 9$

3. Take the square root of both sides. Then $\pm x = \pm 3$.
This looks like four equations (1) $+x = +3$; (3) $-x = +3$.
(2) $+x = -3$; (4) $-x = -3$.

4. $M_{-1}(3)$ Then $+x = -3$. This is equation (2).
$M_{-1}(4)$ Then $+x = +3$. This is equation (1).

So we actually get only two equations, and we get these if we use the + sign on the left in Step 3, and ± on the right.

Rule. To solve an incomplete quadratic equation:

1. *Simplify the equation until it takes the form x^2 = a number.*

2. *Take the square root of both members of the equation, placing the + sign in front of the square root of the left side and the double sign (±) before the principal square root of the right side.*

EXERCISE 71

Find the roots, correct to tenths:

1. $11x^2 - 176 = 0$
2. $5x^2 - 120 = 3x^2 - 22$
3. $\frac{x+2}{x-2} + \frac{x-2}{x+2} = 3$
4. $10c^2 - 2 = 88 - 5c^2$
5. $7(t-3) + t(t-5) = 2t$
6. $\frac{2x-3}{4-x} = \frac{9+x}{3x+2}$
7. Solve for x: $a - 2cx^2 = 3b$.

Solution. 1. $a - 2cx^2 = 3b.$

2. S_a $-2cx^2 = 3b - a.$

3. D_{-2c} $x^2 = \frac{a-3b}{2c}.$

4. $x = \pm\sqrt{\frac{a-3b}{2c}} = \pm\sqrt{\frac{2c(a-3b)}{4c^2}} = \pm\frac{1}{2c}\sqrt{2c(a-3b)}.$

a. Solve each of the following equations for the literal number having exponent 2.

b. Find the value of this number correct to tenths for the given values of the other numbers.

8. (*a*) $A + x^2 = m.$ (*b*) Find x, if $A = 6$ and $m = 90$.
9. (*a*) $cx^2 = d.$ (*b*) Find x, if $c = 12$ and $d = 8$.
10. (*a*) $S = \frac{1}{2}gt^2.$ (*b*) Find t, if $S = 1608$ and $g = 32.16$.
11. (*a*) $f = \frac{mv^2}{r}.$ (*b*) Find v, if $f = 3750$, $r = 5$, and $m = 125$.
12. (*a*) $V = \frac{1}{3}\pi r^2 H.$ (*b*) Find r, if $V = 840$, $\pi = \frac{22}{7}$, and $H = 21$.
13. (*a*) $S = 4\pi r^2.$ (*b*) Find r, if $S = 576$ and $\pi = 3.14$.
14. $3t^2 - \frac{5t^2-6}{2} = \frac{4t^2+1}{3}$
15. $\frac{x^2+x+1}{x-1} - \frac{x^2-x+1}{x+1} = 6$
16. $\frac{8y-1}{4y+1} = \frac{y+1}{3y+1}$
17. $\frac{4}{t+2} + \frac{3}{t-1} = 7$

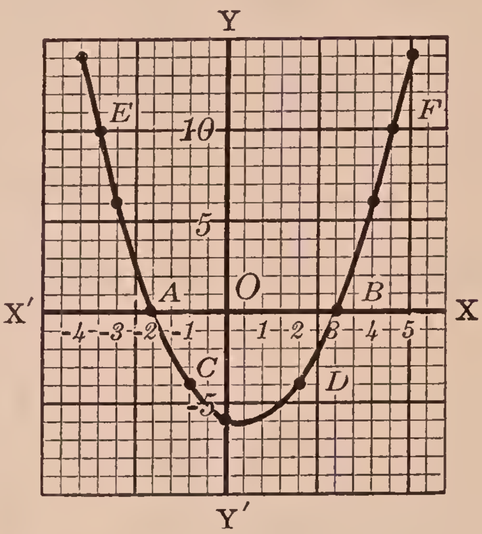
132. Solving a complete quadratic equation graphically.

Example. Solve the equation $\frac{x}{3} - \frac{1}{3} = \frac{2}{x}$.

Solution. 1. M_{3x}. Transpose. $x^2 - x - 6 = 0$.

2. Let $y = x^2 - x - 6$

When x is	Then y is
-4	$+14$
-3	$+6$
-1	-4
0	-6
$+2$	-4
$+4$	$+6$
$+5$	$+14$



3. The graph is above. It is called a **parabola**.

4. y is zero for all points on the x-axis. Therefore the function $x^2 - x - 6 = 0$ at all points where the parabola crosses the x-axis.

At A: $x = -2$. Does $(-2)^2 - (-2) - 6 = 0$? Yes.
At B: $x = +3$. Does $(3)^2 - (3) - 6 = 0$? Yes.
-2 and $+3$ are the roots of $x^2 - x - 6 = 0$

Rule. To solve an equation having one variable graphically:

1. *Clear the equation of fractions; collect like terms; and transpose all terms except zero to the left side of the equation.*

2. *Represent by y the function obtained in Step* 1.

3. *Draw the graph of the function in Step* 1, *making the vertical axis the y-axis or the function-axis.*

4. *At each point where the graph crosses the horizontal axis, y is zero, and the abscissa of the point is a root of the given equation.*

NOTE. The rule furnishes a graphical solution of an equation having one variable not only when the equation is of the second degree, but for an equation of *any degree.*

EXERCISE 72

Solve the following equations graphically.

1. $x^2 + x - 12 = 0$
2. $x^2 - 6x = 16$
3. $x^2 - x = 12$
4. $x^2 = 10 + 3x$
5. $x^2 + x = 20$
6. $2x^2 + 3x = 20$
7. $2x^2 = 7x + 15$
8. $4x^2 - 25 = 0$

Find to the nearest tenth the roots of:

9. $x^2 + 2x - 5 = 0$
10. $x^2 - 3x - 2 = 0$
11. $x^2 - 4x = 2$
12. $x^2 + 3x - 11 = 0$
13. $2x^2 - 3x - 3 = 0$
14. $2x^2 + x = 11$

15. (*a*) Draw the graph of the function $x^2 - 3x$.

(*b*) From this graph determine the roots of the equation $x^2 - 3x = 0$.

(*c*) Determine also, from this graph, the roots of the equation $x^2 - 3x = 4$.

Method. Start at the point $y = 4$, on the y-axis. Imagine a line through this point, parallel to the x-axis. It cuts the graph at two points. The abscissas of these points are the roots of the equation $x^2 - 3x = 4$. Determine them, and check them by substituting them in the equation.

(*d*) Determine from this same graph the real or approximate roots of:

(1) $x^2 - 3x = 10$; (2) $x^2 - 3x = 18$; (3) $x^2 - 3x = 7$.

16. (*a*) Draw the graph of the function $x^2 + 2x$.

(*b*) From this graph, determine the real or approximate roots of the following equations:

(1) $x^2 + 2x = 0$ (2) $x^2 + 2x = 3$ (3) $x^2 + 2x = 8$
(4) $x^2 + 2x - 15 = 0$ (5) $x^2 + 2x = 10$ (6) $x^2 + 2x - 6 = 0$

17. (*a*) Draw the graph of the function $x^2 - 6x$.

(*b*) From this graph, determine the real or approximate roots of:

(1) $x^2 - 6x + 5 = 0$ (2) $x^2 - 6x + 9 = 0$ (3) $x^2 - 6x = 10$

133. Solution of complete quadratics by factoring.

Example. Recall the formula $S = 100\,t - 16\,t^2$ on page 114. How long does it take the ball to rise 100 ft.? That is, find t so that $100 = 100\,t - 16\,t^2$.

Solution. 1. Transpose $16\,t^2 - 100\,t + 100 = 0$
2. D_4 $4t^2 - 25\,t + 25 = 0$
3. Factor. $(4\,t - 5)(t - 5) = 0$
4. If now $4\,t - 5 = 0$, then $(4\,t - 5)(t - 5)$ also $= 0$.
But $4\,t - 5 = 0$, if $4\,t = 5$, or $t = \frac{5}{4}$.
5. Similarly if $t - 5 = 0$, then $(4\,t - 5)(t - 5) = 0$.
But $t - 5 = 0$, if $t = 5$.
These are the roots of the equation as we know from page 115.

134. A fundamental principle used.

If a product is zero, then one or more of its factors is zero.

Thus, if $(x - 1)(x + 2) = 0$, $x - 1 = 0$, or $x + 2 = 0$, or both $x - 1 = 0$ and $x + 2 = 0$. Obviously, if neither is zero, their product is not zero.

Example 2. Solve the equation $M^2 - 2\,M = 35$.

Solution. 1. Transpose $M^2 - 2\,M - 35 = 0.$
2. Factor. $(M - 7)(M + 5) = 0.$
3. $M = +\,7$, and $M = -\,5$.

Note. In Step 3, part of the work was done mentally.

Namely: $(M - 7) = 0$, if $M = 7$; $(M + 5) = 0$, if $M = -\,5$.

Check. These roots check when substituted in the equation.

Also, notice: (*a*) that $(+\,7) + (-\,5) = -\,(-\,2)$ for $+\,7 - 5 = +\,2$, and that $-\,2$ is the coefficient of M.

That is, the product of the roots equals minus the coefficient of the first power of the unknown.

(*b*) that $(+\,7)(-\,5) = -\,35$, the third term of the equation.

That is, the product of the roots equals the term free from the unknown.

This check can be used *only when the coefficient of the squared term is* 1. Here M^2 is $1 \cdot M^2$. If this coefficient is not 1, divide both sides of the equation by it before using this check.

Example 3. Solve the equation $2x^2 - 3ax = 35a^2$.

Solution. 1. Transpose: $2x^2 - 3ax - 35a^2 = 0$

2. Factor. $(2x + 7a)(x - 5a) = 0$

3. $x = -\frac{3}{2}a;\ x = 5a$

Check. If the equation of Step 1 is divided by 2, then

$$x^2 - \frac{3a}{2}x - \frac{35}{2}a^2 = 0.$$

Now, does $\left(-\frac{7}{2}a\right) + (5a) = -\left(-\frac{3a}{2}\right)$?

Does $-\frac{7}{2}a + \frac{10}{2}a = \frac{3a}{2}$? Yes.

Also does $\left(-\frac{7}{2}a\right) \cdot (5a) = -\frac{35}{2}a^2$? Yes.

EXERCISE 73

Solve the following equations by factoring:

1. $y^2 - 12y + 32 = 0$
2. $z^2 + 6z = 55$
3. $m^2 = 63 + 2m$
4. $16x^2 = 1$
5. $x^2 - 11x = 0$
6. $8x^2 + 5x - 3 = 0$
7. $4c^2 - 8c = 21$
8. $3x^2 = \frac{7}{2}x - 1$
9. $\dfrac{4 - x}{1 - x} = \dfrac{12}{3 - x}$
10. $\dfrac{7}{y - 3} - \dfrac{1}{2} = \dfrac{y - 2}{y - 4}$
11. $\dfrac{7}{3 - x} + \dfrac{1}{2} = \dfrac{3}{4 - x}$
12. $\dfrac{2}{3 - x} - \dfrac{6}{8 - x} = 1$

Solve the following equations for x:

13. $x^2 - 2ax - 35a^2 = 0$
14. $4x^2 - 9m^2 = 0$
15. $25x^2 = 16a^2$
16. $2x^2 - 7ax + 3a^2 = 0$
17. $\frac{1}{2}x^2 + ax - \frac{3}{2}a^2 = 0$
18. $\frac{1}{8}x^2 = \frac{5}{2}p^2 - px$
19. $\frac{1}{2}x^2 + \frac{5}{6}rx = \frac{11}{3}r^2$
20. $\frac{6}{5}x^2 + \frac{4}{5}xs = \frac{3}{2}s^2$
21. $\dfrac{3a}{x + 6a} - \dfrac{5}{4} = \dfrac{2a}{x - 5a}$
22. $\dfrac{x + 2c}{x + 3c} + 1 = \dfrac{36c^2}{(x + 3c)^2}$
23. $\dfrac{5m - 2x}{3m - 2x} = \dfrac{6m}{3m - x}$
24. $\dfrac{16t}{x - 6t} = \dfrac{x - 4t}{x - 8t} + 1$

Solution of Quadratics by Completing the Square

135. The need for another method is proved by the attempt to solve the equation $100\,t - 16\,t^2 = 80$.

If you consult the graph on page 115, you find that t must be about 1 and about $5\frac{1}{4}$. Neither of these satisfies the equation, although *they almost satisfy* it. So *the graphical solution is unsatisfactory.*

Simplifying the equation, you get $4\,t^2 - 25\,t + 20 = 0$.

$4\,t^2 - 25\,t + 20$ cannot be factored, so *the factoring method also is not satisfactory.* We must therefore learn a new method.

136. A preliminary review of perfect square trinomials.

Preparation. 1. Write carefully in a column the following expressions and after each its expanded value.

Thus: $(x + 7)^2 = x^2 + 14\,x + 49$. Similarly for:

(*a*) $(x + 4)^2$; (*b*) $(y - 3)^2$; (*c*) $(w + 6)^2$; (*d*) $(z - 7)^2$;
(*e*) $(z + \frac{1}{3})^2$; (*f*) $(w - \frac{1}{5})^2$; (*g*) $(r - \frac{3}{5})^2$; (*h*) $(x - \frac{3}{2})^2$.

2. The result in each case is a *perfect square trinomial.*

3. Write the following expressions in a column; annex to each the necessary third term to make the resulting trinomial a perfect square. Then after it write the square root of it.

Thus: for $x^2 + 8\,x$, write "$x^2 + 8\,x + 16$; $x + 4$."

(*a*) $x^2 + 8\,x$; (*b*) $x^2 - 12\,x$; (*c*) $x^2 - 20\,x$; (*d*) $x^2 - 16\,x$.

In each case, did you take $\frac{1}{2}$ the coefficient of x and square it?

4. Do similarly for : (*a*) $x^2 + \frac{2}{3}\,x$. (How much is $\frac{1}{2}$ of $\frac{2}{3}$?) (*b*) $w^2 + \frac{2}{5}\,w$; (*c*) $y^2 - \frac{4}{3}\,y$; (*d*) $z^2 + \frac{2}{7}\,z$; (*e*) $t^2 - \frac{6}{5}\,t$.

5. Do similarly for: (*a*) $x^2 + \frac{3}{2}\,x$. (How much is $\frac{1}{2}$ of $\frac{3}{2}$?) (*b*) $w^2 + \frac{1}{2}\,w$; (*c*) $x^2 - \frac{1}{3}\,x$; (*d*) $r^2 + \frac{3}{2}\,r$; (*e*) $s^2 - \frac{5}{3}\,s$.

In parts 3 to 5 you have been *completing the square* for *binomials having a squared term with coefficient* 1. To do so, you *add* the *square* of *one half the coefficient of the first degree term.*

137. Solving a quadratic by completing the square.

Example. Solve $x^2 - 12x + 20 = 0$.

Solution. 1. S_{20}: $x^2 - 12x = -20$.

2. Complete the square of the left member by adding 36.

A_{36} $x^2 - 12x + 36 = -20 + 36$

3. $\therefore (x - 6)^2 = 16$

4. Take the square root: $x - 6 = \pm 4$

5. $\therefore x - 6 = +4$, or $x = +10$. This is one root.

And $x - 6 = -4$, or $x = +2$. The other root.

Check. $+10$, and $+2$ satisfy the given equation.

Rule. To solve a quadratic by completing the square.

1. *Simplify the equation and transpose terms until the equation takes the form* $ax^2 + bx = c$.

2. *If a is not 1, divide both sides of the equation by it, so that the equation takes the form* $x^2 + px = q$.

3. *Now take one half of the coefficient of x; square it; add the square to both sides of the equation of Step* 2.

NOTE. In Step 3: to get $\frac{1}{2}$ of a fraction, divide the numerator by 2 or multiply the denominator by 2.

Thus: $\frac{1}{2}$ of $\frac{4}{5}$ is $\frac{2}{5}$; $\frac{1}{2}$ of $\frac{3}{5}$ is $\frac{3}{10}$.

4. *Write the left side as the square of a binomial; express the other side in simplest (fractional) form.*

5. *Take the square root of both sides, writing the double sign,* $\pm$, *before the square root of the right side.*

6. *Let the left square root equal the* $+$ *root of the right side obtained in Step* 5. *Solve the resulting equation.*

7. *Let the left square root equal the* $-$ *root of the right side obtained in Step* 5. *Solve the resulting equation.*

Check by substituting the results of Steps 6 and 7 in the given equation, if they are integers or fractions.

If the results are expressed approximately as decimals, they will not satisfy the given equation. Then (*a*) *the sum of the roots is minus the coefficient of x in the equation of Step* 2.

(*b*) *the product of the roots is the third term in this same equation.*

EXERCISE 74

Solve by completing the square. Check.

1. $x^2 + 8x + 15 = 0$
2. $y^2 - 8y + 12 = 0$
3. $z^2 - 2z - 35 = 0$
4. $x^2 + 2x - 3 = 0$
5. $y^2 - 4y + 5 = 0$
6. $z^2 = 6z + 7$
7. $w^2 - 10w + 9 = 0$
8. $s^2 + 8s - 9 = 0$
9. $x^2 - 5x + 6 = 0$

(Remember $\frac{1}{2}$ of $5 = \frac{5}{2}$; and $(\frac{5}{2})^2 = \frac{25}{4}$.)

10. $x^2 - 3x + 2 = 0$
11. $y^2 + 3y - 4 = 0$
12. $z^2 - 5z = 14$
13. $m^2 + m - 2 = 0$
14. $t^2 + 5t = 6$
15. $w^2 - w = 12$
16. $c^2 - 7c + 10 = 0$
17. $x^2 + 9x = 10$
18. *Illustrative example.* $x^2 + 4x - 4 = 0.$

Solution. 1. $x^2 + 4x = 4$

2. Complete the square: $x^2 + 4x + 4 = 4 + 4$

3. $\therefore (x + 2)^2 = 8$

4. Take the square root: $x + 2 = \pm\sqrt{8}$

5. $\therefore x_1 + 2 = +\sqrt{8}$ | $x_2 + 2 = -\sqrt{8}$

$x_1 = -2 + \sqrt{8}$ | $x_2 = -2 - \sqrt{8}$

Check. Does $(-2 + \sqrt{8}) + (-2 - \sqrt{8}) = -4$? Yes.
Does $(-2 + \sqrt{8})(-2 - \sqrt{8}) = -4$? Does $+4 - 8 = 4$? Yes.

NOTE. For this check, see page 123, end of Rule.

To get the roots correct to tenths, use $\sqrt{8} = 2.828$.

$x_1 = -2 + 2.828$; $\therefore x_1 = .828$, or about .8.

$x_2 = -2 - 2.828$; $\therefore x_2 = -4.828$, or about -4.8.

19. $x^2 + 2x - 4 = 0$
20. $y^2 - 2y - 1 = 0$
21. $z^2 + 4z - 3 = 0$
22. $w^2 - 4w = 6$
23. $t^2 + 6t - 3 = 0$
24. $x^2 - 6x = 1$
25. $x^2 + 3x - 1 = 0$
26. $y^2 - 3y - 2 = 0$
27. $w^2 + w = 5$
28. $z^2 - z = 3$
29. $t^2 - 5t - 7 = 0$
30. $x^2 + 5x - 3 = 0$

EXERCISE 75. *(Optional)*

Illustrative Example. Solve $x^2 + \frac{2}{5}x = \frac{2}{5}$

Solution. 1. $x^2 + \frac{2}{5}x = \frac{2}{5}$

2. $\frac{1}{2}$ of $\frac{2}{5} = \frac{1}{5}$; $(\frac{1}{5})^2 = \frac{1}{25}$

3. $A(\frac{1}{25})$ $x^2 + \frac{2}{5}x + \frac{1}{25} = \frac{2}{5} + \frac{1}{25}$.

4. $\therefore (x + \frac{1}{5})^2 = \frac{11}{25}$

5. $\therefore x + \frac{1}{5} = \pm \frac{1}{5}\sqrt{11}$

6. $\therefore x_1 = -\frac{1}{5} + \frac{1}{5}\sqrt{11}$ | $x_2 = -\frac{1}{5} - \frac{1}{5}\sqrt{11}$

$x_1 = \frac{-1 + \sqrt{11}}{5}$ | $x_2 = \frac{-1 - \sqrt{11}}{5}$

7. $\therefore x_1 = \frac{-1 + 3.3166}{5}$ | $x_2 = \frac{-1 - 3.3166}{5}$

$x_1 = \frac{2.3166}{5}$; or $x_1 = .46^+$ | $x_2 = \frac{-4.3166}{5}$; or $x_2 = -.86^+$

Solve correct to hundredths.

1. $x^2 + \frac{2}{3}x - 7 = 0$

2. $x^2 - \frac{4}{3}x + \frac{1}{3} = 0$

3. $x^2 - \frac{2}{5}x - \frac{3}{5} = 0$

4. $x^2 + \frac{8}{3}x - 2 = 0$

5. $y^2 - \frac{6}{5}y - \frac{1}{5} = 0$

6. $z^2 - \frac{10}{3}z - 1 = 0$

7. $x^2 + \frac{1}{2}x - \frac{1}{2} = 0$

(*Hint.* $\frac{1}{2}$ of $\frac{1}{2} = \frac{1}{4}$.)

8. $x^2 - \frac{3}{4}x = 1$

9. $y^2 + \frac{3}{5}y - 3 = 0$

10. $x^2 - \frac{5}{2}x + \frac{1}{2} = 0$

11. $z^2 + \frac{7}{3}z + \frac{1}{3} = 0$

12. Solve the equation $3x^2 - 5x + 2 = 0$.

Solution. 1. $3x^2 - 5x + 2 = 0$.

2. D_3 $x^2 - \frac{5}{3}x + \frac{1}{3} = 0$.

(Complete the solution.)

13. $2x^2 - 5x + 2 = 0$

14. $3x^2 + 10x + 3 = 0$

15. $6x^2 + x - 2 = 0$

16. $4p^2 - 2p = 1$

17. $5r^2 + 2r = 7$

18. $3m^2 + 5m - 2 = 0$

19. $3x^2 - 2x - 3 = 0$

20. $8y^2 + 4y - 1 = 0$

21. $4w^2 - 3w - 2 = 0$

22. $6t^2 + 8t - 5 = 0$

23. $2s^2 - 10s - 9 = 0$

24. $10w^2 + 4w - 3 = 0$

Solution of Quadratic Equation by a Formula

138. All quadratic equations having one unknown can be put in the form $ax^2 + bx + c = 0.$

If we solve $ax^2 + bx + c = 0$ for x, the roots may be used as formulas for the roots of any quadratic equation.

Solution. 1. $$ax^2 + bx + c = 0.$$

2. D_a $$x^2 + \frac{b}{a} \cdot x + \frac{c}{a} = 0.$$

3. $S_{\frac{c}{a}}$ $$x^2 + \frac{b}{a} \cdot x = -\frac{c}{a}.$$

4. $$\frac{1}{2} \text{ of } \left(\frac{b}{a}\right) = \frac{b}{2a}; \quad \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}.$$

5. $A_{\left(\frac{b^2}{4a^2}\right)}$ $$x^2 + \frac{b}{a} \cdot x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$$

6. $$\therefore \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

7. $$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

8. $$\therefore x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

9. $$\therefore \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Example. Solve $2x^2 - 3x - 5 = 0$, by the formula.

Solution. 1. Comparing this equation with $ax^2 + bx + c = 0$,
$$a = 2, b = -3, c = -5.$$

2. The formula is $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

3. Substituting, $$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2 \cdot 2}.$$

4. $$\therefore x = \frac{+3 \pm \sqrt{9 + 40}}{4}, \text{ or } \frac{+3 \pm 7}{4}.$$

5. $$\therefore x_1 = \frac{3 + 7}{4} = \frac{10}{4} = \frac{5}{2}; \text{ and } x_2 = \frac{3 - 7}{4} = \frac{-4}{4} = -1.$$

139. The sum and the product of the roots of a quadratic equation have been used to check equations. We shall now prove the correctness of the rules used. The general quadratic equation is

$$ax^2 + bx + c = 0.$$

In § 138 you proved that the roots of this equation are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad \text{and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

I. $$\therefore r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$\boxed{\therefore r_1 + r_2 = \frac{-2b}{2a}, \text{ or } -\frac{b}{a}}$$

II. $$r_1 \cdot r_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$\therefore r_1 r_2 = \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2}$$

$$\boxed{\therefore r_1 r_2 = \frac{b^2 - (b^2 - 4ac)}{4a^2} \text{ or } \frac{a}{c}} \quad \frac{c}{a}$$

Stated in words: *for the quadratic* $ax^2 + bx + c = 0$, *the sum of the roots is* $-\frac{b}{a}$, *and the product is* $\frac{c}{a}$.

Example. What are the sum and the product of the roots of

$$2x^2 = 9x + 5?$$

Solution. 1. $$2x^2 = 9x + 5.$$

2. $$\therefore 2x^2 - 9x - 5 = 0.$$

3. $$\therefore a = 2; \quad b = -9; \quad c = -5.$$

4. $$\therefore r_1 + r_2 = -\frac{b}{a} = -\left(\frac{-9}{2}\right) = \frac{9}{2}.$$

5. $$\therefore r_1 r_2 = \frac{c}{a} = \frac{-5}{2} = -\frac{5}{2}.$$

NOTE. Always arrange the quadratic as in Step 2, with all terms on the left side, and make the coefficient of x^2 positive.

140. The example at the bottom of page 126 shows how important the **quadratic formula is.** *You can and should memorize the general equation* $ax^2 + bx + c = 0$ *and the formula in two minutes.*

Notice that we were able to get this formula because we know how to "complete the square," so *that method* also is important. *However,* from now on, when you cannot solve a quadratic quickly by factoring, solve it at once by the formula.

Remember that there are two roots. Pattern your solution after that at the bottom of page 126.

EXERCISE 76

Find the roots correct to tenths; check by using the conclusion stated on page 127.

1. $x^2 + 7x + 6 = 0$
2. $x^2 - 2x - 15 = 0$
3. $2x^2 - 5x + 2 = 0$
4. $3y^2 + 7y + 2 = 0$
5. $2z^2 - 7z + 3 = 0$
6. $2w^2 - 3w - 2 = 0$
7. $x^2 + 2x - 35 = 0$
8. $12x^2 - 5x - 2 = 0$
9. $10m^2 + 11m - 6 = 0$
10. $24x^2 - 14x - 3 = 0$
11. $6m^2 = 7m + 3$
12. $8x^2 + 2x = 1$
13. $2w^2 - w = 1$
14. $4 - x = 5x^2$
15. $0 = 3y^2 - 20 - 7y$
16. $x^2 + 2x - 2 = 0$
17. $x^2 - 2x - 1 = 0$
18. $2x^2 - 3x - 1 = 0$
19. $3y^2 - 4y = 1$
20. $2z^2 = 8z + 1$
21. $4t^2 + 9t + 2 = 0$
22. $5c^2 - 4c = 2$
23. $3a^2 = 4 - 6a$
24. $3c^2 - 2 = 7c$
25. $x + 5 = 3x^2$
26. $x^2 - .5x = .5$
27. $x^2 - .7x - .3 = 0$
28. $y^2 + .6y - .4 = 0$
29. $z^2 - 2.5z - 1.5 = 0$
30. $w^2 + .3w - 6.8 = 0$
31. $2x^2 - 3x + d = 0$
32. $3x^2 + cx + 5 = 0$
33. $rx^2 - 2x + t = 0$
34. $2mx^2 + 3x - 4a = 0$

141. When **solving fractional equations** we often get a quadratic. A difficulty, often met, is illustrated by the following example.

Example. Solve $\frac{x+3}{x^2-1}+\frac{x-3}{x^2-x}+\frac{x+2}{x^2+x}=0$

After solving this equation it appears that $x = 1$, or $-\frac{5}{3}$.

When 1 is substituted for x, does $\frac{4}{0}+\frac{-2}{0}+\frac{3}{2}=0$?

But $\frac{4}{0}$, and $\frac{-2}{0}$ do not have any meaning. (§ 20.) From this we conclude that 1 cannot be a root of the given equation.

The value $-\frac{5}{3}$ does not cause this same trouble.

So, when solving a fractional equation, an *apparent root must be rejected if it makes a denominator have the value zero.*

EXERCISE 77

Express surd roots correct to tenths.

1. $\frac{3}{5x^2}-\frac{2}{x}-5=0$
2. $\frac{4}{x}-\frac{5}{2x+3}=3$
3. $\frac{3}{y+8}-\frac{4}{y-2}=2$
4. $\frac{2}{t-1}-\frac{1}{3t+1}=\frac{1}{2}$
5. $\frac{2}{z-4}-\frac{1}{z-2}=2$
6. $\frac{7}{2x-3}=\frac{x-1}{x-2}+\frac{1}{2}$
7. $c-\frac{3}{2c-3}=0$
8. $\frac{3}{3r+1}+\frac{4}{3r}=-2$
9. $\frac{s}{s+3}+\frac{s}{6}=1$
10. $\frac{c}{2c+5}-\frac{1}{2c-3}=1$
11. $1+\frac{2}{2p-3}=\frac{3}{p-4}$
12. $\frac{m}{2m+1}-\frac{m}{3m-1}=1$
13. $\frac{2}{1-4y}=\frac{1}{5}-\frac{3}{1+2y}$
14. $\frac{2x-1}{x+1}-\frac{2x+1}{x-1}=3$
15. $\frac{9w-2}{5}-\frac{w-1}{w+1}=1$
16. $\frac{14-3m}{4}-\frac{6+m}{4-m}=5$

EXERCISE 78

1. If 24 be added to 5 times a certain number, the result equals the square of the number. What is the number?

2. If twice the square of a certain number be diminished by the number itself, the result is 15. What is the number?

3. There are two consecutive even integers whose product is 624. What are they?

4. There are three consecutive odd integers such that the square of the first, increased by the product of the other two, is 844. What are they?

5. The sum of a certain number and its reciprocal is $\frac{113}{56}$. What is the number? (*Hint.* The reciprocal of x is $\frac{1}{x}$.)

6. The denominator of a certain fraction exceeds its numerator by 5. The sum of the fraction and its reciprocal is $\frac{97}{36}$. What is the fraction?

7. The base of a certain triangle exceeds its altitude by 7 inches. Its area is 99 square inches. What are its dimensions?

8. The area of a certain rectangle is 405 square feet. The sum of its base and altitude is 42 feet. What are its dimensions?

9. The total area of certain two squares is 765 square inches. The side of one exceeds the side of the other by 3 inches. How long is the side of each?

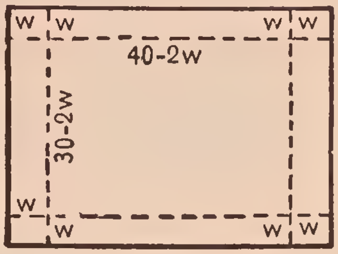
10. The area of the main waiting room of the Union Railway Station at Washington, D. C., is 28,600 square feet. The sum of its length and width is 350 feet. What are its dimensions?

11. The altitude of a certain right triangle is 6 feet more than its base. The hypotenuse is 30 feet. What are its base and altitude?

12. If a certain number be increased by twice its reciprocal, the sum is $3\frac{3}{10}$. What is the number?

13. A lawn is 30 by 40 feet. How wide a strip must be cut around it when mowing the grass to have cut half of it?

Hint. Referring to the figure, it is clear that if $w =$ the number of feet in the width of the border cut, then the dimensions of the uncut part of the lawn are $(30 - 2w)$ and $(40 - 2w)$.



Hence, $(30 - 2w)(40 - 2w) = \frac{1}{2} \cdot 30 \cdot 40.$

$\therefore 2(15 - w) \cdot 2 \cdot (20 - w) = \frac{1}{2} \cdot 30 \cdot 40.$

14. A boy is plowing a field whose dimensions are 20 rods and 40 rods. How wide a border must be cut around it in order to have completed $\frac{2}{3}$ of his plowing?

15. The numerator of a certain fraction is 3 less than its denominator. If 4 be added to both numerator and denominator, the new fraction exceeds the old by $\frac{1}{8}$. What is the fraction?

16. A man traveled 75 miles by automobile at a certain rate. By increasing his average rate 5 miles per hour, he made the return trip in $\frac{3}{4}$ hour less time than the time going. What was the rate going and returning?

17. Suppose a motor boat, traveling at the rate of 12 miles per hour in still water, goes 30 miles downstream and back again in a total of $5\frac{1}{2}$ hours. What is the rate of the current of the stream to the nearest tenth of a mile?

18. A crew can row 8 miles downstream, and back again, in $4\frac{4}{5}$ hours. If the rate of the stream is 4 miles per hour, what is the rate of the crew in still water?

19. The denominator of a certain fraction exceeds its numerator by 3. If the numerator be trebled, and the denominator be doubled, the sum of the resulting fraction and the original fraction is 1. What is the fraction?

20. Some boys are canoeing on a river, in part of which the current is 5 miles an hour, and in another part 3 miles an hour. If, when going downstream, they go 4 miles where the current is rapid and 8 miles where it is less rapid in a total time of $1\frac{5}{6}$ hours, what is their rate in still water?

Note. Additional problems appear on Page 248.

Imaginary Numbers and Roots of a Quadratic

142. A new difficulty and a new kind of number.

Example 1. Solve the equation $x^2 - 2x + 5 = 0$.

Solution. 1. Use the formula method of solution.

$$a = 1; \quad b = -2; \quad c = 5.$$

2. $$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1}$$

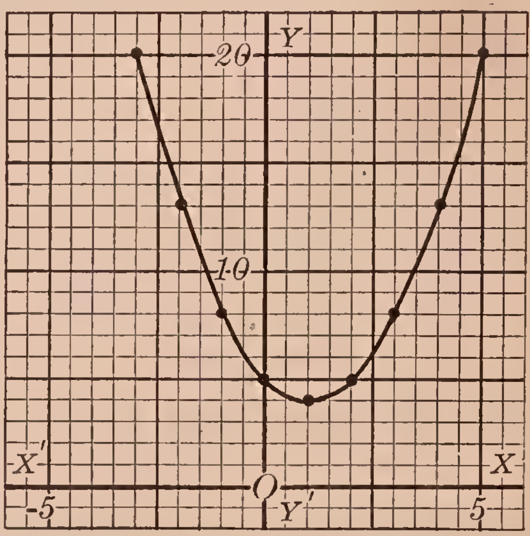
3. $$= \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm \sqrt{-16}}{2}$$

4. What does $\sqrt{-16}$ mean?

-4 is not the square root of -16, since $(-4)^2 = +16$.
$+4$ is not the square root of -16, since $(+4)^2 = +16$.

No number, studied so far, will produce a negative result when it is squared, so we have met a new difficulty. Let us try to solve the equation graphically.

When x is	Then y is
-3	$+20$
-2	$+13$
-1	$+8$
0	$+5$
$+1$	$+4$
$+3$	$+8$
$+2$	$+5$
$+4$	$+13$
$+5$	$+20$



According to the rule on page 118, the roots of the equation are the abscissas of the points where the parabola crosses the x-axis. *But this parabola does not cross the x-axis.* This confirms our realization that a new difficulty is introduced by this equation.

The manner in which mathematicians finally overcame this difficulty is taught on pages 133 to 136.

143. **Mathematicians settled this difficulty by inventing a new kind of number** — *an imaginary number* they called it.

An **imaginary number** is the indicated square root of a negative number, as $\sqrt{-6}$; $\sqrt{-3}$; $\sqrt{-\frac{1}{2}}$.

The numbers studied previously, in contrast to the imaginary numbers, are called **real numbers.**

NOTE. Negative numbers were once called *numerae fictae,* — fictitious numbers. Both kinds are now very real to all scientists.

144. **Every imaginary number can be expressed as the product of a real number and $\sqrt{-1}$.**

$\sqrt{-1}$ is indicated by i, and is called the **imaginary unit.**

Thus,
$$\sqrt{-16} = \sqrt{16(-1)} = 4 \cdot \sqrt{-1} = 4i$$
$$\sqrt{-a^2} = \sqrt{a^2(-1)} = \sqrt{a^2} \cdot \sqrt{-1} = ai$$
$$\sqrt{-5} = \sqrt{5(-1)} = \sqrt{-1} \cdot \sqrt{5} = i\sqrt{5}$$
$$\frac{5}{2} \pm \sqrt{-\frac{27}{4}} = \frac{5}{2} \pm \sqrt{\frac{27}{4}(-1)} = \frac{5}{2} \pm \frac{3i}{2}\sqrt{3} = \frac{5 \pm 3i\sqrt{3}}{2}$$

The numerator of this result is a **complex number.**

EXERCISE 79

Express in terms of the unit i:

1. $\sqrt{-4}$
2. $\sqrt{-9}$
3. $\sqrt{-169}$
4. $\sqrt{-81}$
5. $\sqrt{-64}$
6. $\sqrt{-196}$
7. $\sqrt{-12}$
8. $\sqrt{-27}$
9. $\sqrt{-32}$
10. $\sqrt{-63}$
11. $\sqrt{-48}$
12. $\sqrt{-75}$
13. $\sqrt{-\frac{1}{9}}$
14. $\sqrt{-\frac{1}{16}}$
15. $\sqrt{-\frac{9}{25}}$
16. $\sqrt{-\frac{5}{9}}$
17. $\sqrt{-\frac{3}{16}}$
18. $\sqrt{-\frac{7}{4}}$
19. $\sqrt{-\frac{2}{3}}$
20. $\sqrt{-\frac{4}{5}}$
21. $\sqrt{-\frac{1}{6}}$
22. $\sqrt{-\frac{3}{8}}$
23. $\sqrt{-\frac{5}{12}}$
24. $\sqrt{-\frac{3}{10}}$
25. $\frac{1}{3} \pm \sqrt{-\frac{5}{9}}$
26. $\frac{3}{2} \pm \sqrt{-\frac{3}{4}}$
27. $\frac{2}{5} \pm \sqrt{-\frac{6}{25}}$
28. $-\frac{5}{4} \pm \sqrt{-\frac{9}{16}}$
29. $-\frac{3}{8} \pm \sqrt{-\frac{27}{64}}$
30. $-\frac{2}{7} \pm \sqrt{-\frac{12}{49}}$
31. $-\frac{5}{6} \pm \sqrt{-\frac{7}{36}}$
32. $\frac{9}{10} \pm \sqrt{-\frac{27}{100}}$
33. $\frac{5}{9} \pm \sqrt{-\frac{8}{81}}$

145.* **Adding and subtracting imaginary and complex numbers.**

Just as $3 \cdot 1 + 5 \cdot 1 = 8 \cdot 1$, or 8, so $3 \cdot i + 5\,i = 8\,i$; similarly, $(2 + 5\,i) - (3 - 4\,i) = 2 + 5\,i - 3 + 4\,i = 9\,i - 1$.

146. Powers of the imaginary unit ***i*****.**

Since $i = \sqrt{-1}$, then $i^2 = -1$.

$$i^3 = i^2 \cdot i = -1 \cdot i. \qquad \therefore\ i^3 = -i.$$

$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1. \qquad \therefore\ i^4 = 1.$$

Similarly: $i^5 = i$; $i^6 = -1$; $i^7 = -i$; $i^8 = +1$.

147. Multiplication of imaginary and complex numbers.

Example 1. $\sqrt{-2} \cdot \sqrt{-3} = i\sqrt{2} \cdot i\sqrt{3} = i^2\sqrt{6} = -6$.

Example 2. Find $(2 - \sqrt{-3})(5 + \sqrt{-3})$.

Solution. 1. $(2 - \sqrt{-3})(5 + \sqrt{-3}) = (2 - i\sqrt{3})(5 + i\sqrt{3})$

2. $= 10 - 3\,i\sqrt{3} - i^2 \cdot 3$

3. $= 10 - 3\,i\sqrt{3} - (-1) \cdot 3$

4. $= 13 - 3\,i\sqrt{3}$.

EXERCISE 80

Find the following products:

1. $\sqrt{-4} \cdot \sqrt{-16}$
2. $\sqrt{-2} \cdot \sqrt{-18}$
3. $\sqrt{-9} \cdot \sqrt{-2}$
4. $\sqrt{-5} \cdot \sqrt{-10}$
5. $\sqrt{-25\,x^2} \cdot \sqrt{-4\,x^2}$
6. $3\sqrt{-5} \cdot 3\sqrt{-5}$
7. $4\sqrt{-3} \cdot 6\sqrt{-3}$
8. $x\sqrt{-y} \cdot x\sqrt{-y}$
9. $a\sqrt{-b} \cdot c\sqrt{-d}$
10. $\sqrt{-xy} \cdot (-\sqrt{-ab})$
11. $(3 + \sqrt{-2})(3 - \sqrt{-2})$
12. $(2 - \sqrt{-6})(2 + \sqrt{-6})$
13. $(2 + \sqrt{-5})(2 + 3\sqrt{-5})$
14. $(8 - \sqrt{-7})(10 + \sqrt{-7})$
15. $(-2 + \sqrt{-2})(-2 - \sqrt{-2})$
16. $(6 - \sqrt{-3})^2$
17. $(a + \sqrt{-b})^2$
18. $\{\frac{1}{3}(-1 + \sqrt{-2})\}^2$
19. $\{\frac{1}{2}(-1 - \sqrt{-3})\}^2$

* Note to the Teacher. Pages 134–135 can be omitted.

148. Division of imaginary and complex numbers.

$$\frac{10}{\sqrt{-2}} = \frac{10}{i\sqrt{2}} = \frac{10 \cdot i\sqrt{2}}{i\sqrt{2} \cdot i\sqrt{2}} = \frac{10\,i\sqrt{2}}{2\,i^2} = \frac{5\,i\sqrt{2}}{-1} = -5\,i\sqrt{2}$$

To rationalize a denominator multiply it by itself.

EXERCISE 81

Express the quotient with rational denominators.

1. $\frac{\sqrt{-36}}{\sqrt{-6}}$ 3. $\frac{\sqrt{-40\,x}}{\sqrt{-5\,x}}$ 5. $\frac{15}{\sqrt{-3}}$ 7. $\frac{\sqrt{10}}{\sqrt{-2}}$

2. $\frac{\sqrt{-32}}{8}$ 4. $\frac{6}{\sqrt{-2}}$ 6. $\frac{14}{\sqrt{-2}}$ 8. $\frac{\sqrt{28}}{\sqrt{-14}}$

9. *Illustrative example.* $\frac{2}{1+\sqrt{-3}} = \frac{2}{1+i\sqrt{3}}$

$$= \frac{2(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{2(1-i\sqrt{3})}{1-3\,i^2}, \text{ etc.}$$

NOTE. $(1-i\sqrt{3})$ and $(1+i\sqrt{3})$ are **conjugate complex numbers.**

To rationalize a complex denominator, multiply the terms of the fraction by the conjugate of the denominator.

10. $\frac{3}{1+\sqrt{-3}}$ 13. $\frac{5}{1+\sqrt{-2}}$ 16. $\frac{3-\sqrt{-2}}{3+2\sqrt{-2}}$

11. $\frac{13}{3-\sqrt{-2}}$ 14. $\frac{1-\sqrt{-1}}{1+\sqrt{-1}}$ 17. $\frac{5+\sqrt{-3}}{2-\sqrt{-3}}$

12. $\frac{4}{2-\sqrt{-2}}$ 15. $\frac{1-\sqrt{-1}}{2+\sqrt{-1}}$ 18. $\frac{4+\sqrt{-2}}{5-2\sqrt{-2}}$

19. Find the value of $x^2 + x - 1$ when $x = 1 + \sqrt{-2}$

20. Find the value of $\frac{x+1}{x-1}$ when $x = 2 + \sqrt{-3}$

21. Find the value of $x^2 - 2\,x + 3$ when $x = 1 - \sqrt{-2}$

22. Find the value of $\frac{x}{3-x}$ when $x = 3 - \sqrt{-5}$

149. Solving quadratics which have imaginary roots.

Example. Solve the equation $x^2 + x + 2 = 0$.

Solution. 1. $a = 1;\ b = 1;\ c = 2.$

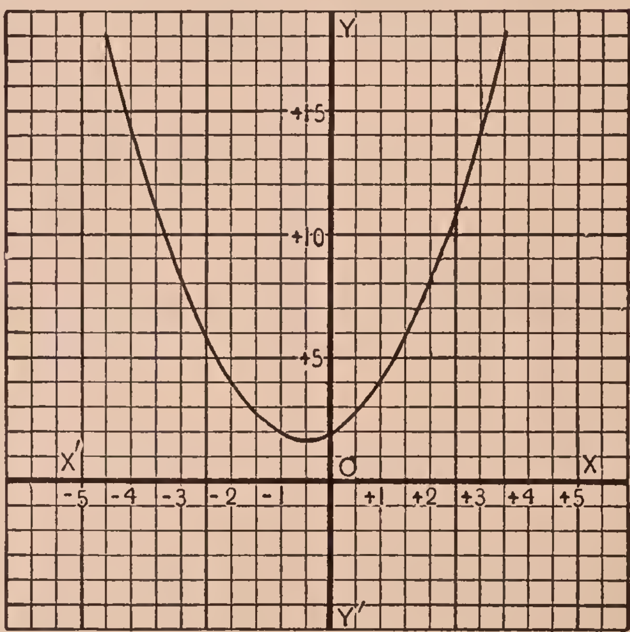
2. $\therefore x = \dfrac{-1 \pm \sqrt{1-8}}{2} = \dfrac{-1 \pm \sqrt{-7}}{2} = \dfrac{-1 \pm i\sqrt{7}}{2}.$

3. $\therefore x_1 = \dfrac{-1 + i\sqrt{7}}{2};\quad x_2 = \dfrac{-1 - i\sqrt{7}}{2}.$

We shall also solve this equation graphically.

Solution. 1. Let $y = x^2 + x + 2$.

When $x =$	0	+ 1	+ 2	+ 3	− 1	− 2	− 3	− 4
then $y =$	+ 2	+ 4	+ 8	+ 14	+ 2	+ 4	+ 8	+ 14



3. The graph has the same shape as the graphs obtained when solving other quadratic equations; *but the graph does not cross* the horizontal axis at all. Hence, y or $x^2 + x + 2$ is never zero for any real value of x.

Remember this is characteristic of the graph of a quadratic which has *imaginary* roots.

EXERCISE 82

Solve the following equations either by completing the square or by the formula. Draw the graphs of the first three equations.

1. $x^2 + x + 3 = 0$
2. $x^2 + 2x + 2 = 0$
3. $x^2 - 3x + 4 = 0$
4. $2x^2 - x + 1 = 0$
5. $3y^2 - 2y + 2 = 0$
6. $2a^2 - 3a + 2 = 0$
7. $3t^2 - 4t + 2 = 0$
8. $4r^2 - 5r + 2 = 0$
9. $7s^2 + 6s + 3 = 0$
10. $6c^2 - 5c + 2 = 0$

150. The **numbers considered in this text** so far are:

I. The real numbers, which include:

(*a*) the rational numbers, which are the positive and negative integers, and the fractions whose terms are such integers;

(*b*) the irrational numbers, which, thus far, are the quadratic surds and surd expressions. (See p. 105.)

II. The imaginary and complex numbers. (See p. 133.)

Historical Note. Greek mathematicians as early as Euclid were able to solve certain quadratics by a geometric method, about which the student may learn when he studies plane geometry. Heron of Alexandria, about 110 B.C., proposed a problem which leads to a quadratic. His solution is not given, but his result would indicate that he probably solved the equation by a rule which might be obtained from the quadratic by completing its square in a certain manner. Diophantus, 275 A.D., gave many problems which lead to quadratic equations. He considered three separate kinds of quadratics. He gave only one root for a quadratic, even when the equation had two roots.

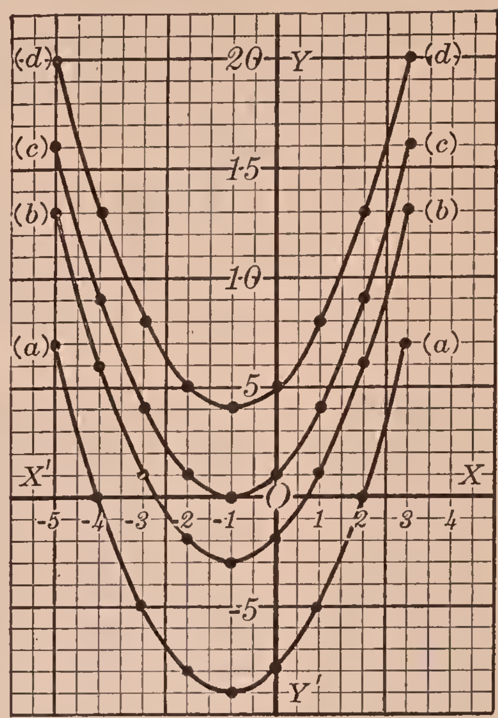
The Hindu mathematicians, knowing about negative numbers, considered one general quadratic. The Hindus knew that a quadratic has two roots, but they usually rejected any negative roots.

The Arabians went back to the practice of Diophantus in considering three or more kinds of quadratics. Mohammed Ben Musa, 820 A.D., had five kinds. He admitted two roots when both were positive. Alkarchi gave a purely algebraic solution of a quadratic by completing the square, and refers to this method as being a diophantic method.

In Europe, mathematicians followed the practice of the Arabians, and by the time of Widmann, 1489, had twenty-four special forms of equations. These were solved by rules which were learned and used in a mechanical manner. Stifel, 1486–1567, finally brought the study of quadratics back to the point that had been reached by the Hindus one thousand years before. He gave only three normal forms for the quadratic; he allowed double roots when they were both positive. Stevin, 1548–1620, went still farther. He gave only one normal form for the general quadratic, as do we; he solved this in both a geometric and an algebraic manner, giving the method of completing the square. He allowed negative roots.

151. The different kinds of roots of a quadratic.

Consider the following equations, their roots and graphs.



(a) $x^2 + 2x - 8 = 0$

$$x = \frac{-2 \pm \sqrt{36}}{2} = \frac{-2 \pm 6}{2}$$

$x_1 = -4;\ x_2 = +2$

(b) $x^2 + 2x - 2 = 0$

$$x = \frac{-2 \pm \sqrt{12}}{2} = \frac{-2 \pm 2\sqrt{3}}{2}$$

$x_1 = -1 - \sqrt{3}$, or -1.7

$x_2 = -1 + \sqrt{3}$, or $+.7$

(c) $x^2 + 2x + 1 = 0$

$$x = \frac{-2 \pm \sqrt{4 - 4}}{2}$$

$x_1 = \dfrac{-2 - 0}{2}$, or -1 $\qquad x_2 = \dfrac{-2 + 0}{2}$, or -1

(d) $x^2 + 2x + 5 = 0$

$$x = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$x_1 = \dfrac{-2 - 4i}{2}$, or $-1 - 2i$

$x_2 = \dfrac{-2 + 4i}{2}$, or $-1 + 2i$

In	The roots are	$b^2 - 4ac$ is	The graph
Equation (*a*)	Real Rational Unequal	Positive and a perfect square	Crosses the x-axis at two points
Equation (*b*)	Real Irrational Unequal	Positive and not a perfect square	Crosses the x-axis at two points
Equation (*c*)	Real Rational Equal	Zero	Touches the x-axis at a point
Equation (*d*)	Imaginary Unequal	Negative	Is entirely above the x-axis

152. **It is possible to determine the kind of roots of a quadratic equation** (with rational coefficients) **without first finding the roots.**

The roots of $ax^2 + bx + c = 0$ are, as you know,

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

The *kind* of roots is determined by the value of $b^2 - 4ac$ as you have seen on page 138. For this reason, $b^2 - 4ac$ is called the **discriminant of the quadratic.**

Rules. 1. *If $b^2 - 4ac$ is positive, the roots are real and unequal. They are rational if $b^2 - 4ac$ is a perfect square; they are irrational if $b^2 - 4ac$ is positive and not a perfect square.*

2. *If $b^2 - 4ac$ = zero, the roots are real, rational, and equal.*

3. *If $b^2 - 4ac$ is negative, the roots are imaginary or complex numbers.*

Example 1. *By inspection*, determine the kind of roots of

$$2x^2 - 5x - 18 = 0.$$

Solution. 1. $b^2 - 4ac = (-5)^2 - 4(2)(-18).$

2. $\therefore b^2 - 4ac = 25 + 144 = 169$, or 13^2.

3. $\therefore$ by Rule 1, the roots are real, rational, and unequal.

Example 2. What kind of roots has $3x^2 + 2x + 1 = 0$?

Solution. 1. $b^2 - 4ac = (2)^2 - 4(3)(1)$, or -8.

2. $\therefore$ by Rule 3, the roots are complex numbers.

EXERCISE 83

Determine by inspection the kind of roots of:

1. $y^2 - 7y + 12 = 0$

2. $9x^2 - 24x + 16 = 0$

3. $2m^2 - m + 1 = 0$

4. $3y^2 = 5y + 2$

5. $7x^2 + 5x = 9$

6. $1 + 6t + 4t^2 = 0$

7. $3x - 9x^2 + 1 = 0$

8. $x^2 = 2x - 7$

9. $10 = 4x + 5x^2$

10. $16m^2 - 4 = 0$

11. $9x^2 - 30x + 25 = 0$

12. $x^2 - 4x + 8 = 0$

EXERCISE 84

Find the sum and the product of the roots of the following equations without solving the equations. (See p. 127)

1. $x^2 + 8x + 7 = 0$
2. $z^2 - 9z + 14 = 0$
3. $8x^2 - 2x - 15 = 0$
4. $63y^2 - 16y + 1 = 0$
5. $4a = 7 - 3a^2$
6. $x^2 + 5bx = -4b^2$
7. $15x^2 - a^2 = 2ax$
8. $3x^2 = 4xt + 2t^2$

9. One root of $3x^2 - 2x - 5 = 0$ is -1. Find the other root. *Suggestion.* What is the sum of the roots?

10. One root of $2x^2 - 7x + 3 = 0$ is 3. Find the other.

11. One root of $6x^2 - 1 = 5x$ is $-\frac{1}{6}$. Find the other root.

12. Find k so that one root of $x^2 + 4x - k = 0$ will be -12.

Solution. 1. $r_1 + r_2 = -4$. $\therefore -12 + r_2 = -4$. $\therefore r_2 = 8$.
2. But $r_1r_2 = -k$. $\therefore -k = (-12)\cdot 8$; $\therefore k = ?$

Check. Substitute in $x^2 + 4x - k = 0$ the value of k you find; solve the equation, and thus determine whether -12 really is one root.

13. Find k so that $-\frac{2}{3}$ may be a root of $6x^2 + 7x + 2k = 0$.

14. Find k so that $\frac{3}{4}$ may be a root of $8x^2 - 10x = -k$.

15. Find a so that 7 may be a root of $7z^2 - 48z - a = 0$.

16. Find n so that $\frac{2}{3}$ will be a root of $3x^2 + nx = 4$.

17. Find p so that $-\frac{3}{5}$ will be a root of $10x^2 - 5 = px$.

18. Find c so that the roots of $2z^2 - 8z + c = 0$ shall be equal.

Solution. 1. Let the two equal roots be represented by r.

2. $\therefore r + r = ?$ $\therefore r = ?$ But $r \cdot r = \frac{c}{2}$. $\therefore c = ?$

Or the example can be solved by using Rule 2, § 152.

Find:

19. t so that the roots of $3x^2 - 4x + t = 0$ shall be equal.

20. s so that the roots of $4x^2 - sx + 9 = 0$ shall be equal.

NOTE. Additional examples appear on page 249.

153. Forming quadratic equations having given roots. There are two methods.

Example 1. Form the equation whose roots are $\frac{1}{2}$ and $-\frac{3}{4}$.

Solution. 1. If $x = \frac{1}{2}$, then $x - \frac{1}{2} = 0$.

2. If $x = -\frac{3}{4}$, then $x + \frac{3}{4} = 0$.

3. $\therefore (x - \frac{1}{2})(x + \frac{3}{4}) = 0$, or $x^2 + \frac{1}{4}x - \frac{3}{8} = 0$.

4. $\therefore 8x^2 + 2x - 3 = 0$.

Check by solving the equation.

NOTE. This method may be used when forming an equation which shall have three or more given roots. Use this method in the even examples below.

Example 2. Form the equation whose roots are 2.4 and $-.5$.

Solution. 1. Let the coefficient of x^2 be 1. Then the equation is

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0. \qquad \text{(See § 139)}$$

2. But $r_1 + r_2 = 2.4 + (-.5)$, or 1.9;

and $r_1 \cdot r_2 = 2.4 \cdot (-.5)$, or -1.2.

3. $\therefore$ the equation is $x^2 - 1.9x - 1.2 = 0$.

Check by solving the equation.

EXERCISE 85

Form the equation whose roots shall be

1. $2, 5$
2. $-3, 1$
3. $6, -1$
4. $-3, -7$
5. $\frac{1}{2}, \frac{3}{4}$
6. $\frac{2}{3}, -\frac{1}{6}$
7. $1.5, 2$
8. $.2, .5$
9. $1.5, -2.4$
10. $-3.6, -.25$
11. $+3m, +2m$
12. $-4t, +7t$
13. $\frac{1}{2}c, -\frac{2}{3}c$
14. $d + e, d - e$
15. $\sqrt{5}, -\sqrt{5}$
16. $1 + \sqrt{2}, 1 - \sqrt{2}$
17. $\sqrt{3} + 2, \sqrt{3} - 2$
18. $3 + \sqrt{6}, 3 - \sqrt{6}$
19. $b + 2c, b - 2c$
20. $\sqrt{m}, -\sqrt{m}$
21. $a + \sqrt{b}, a - \sqrt{b}$
22. $\frac{1 + \sqrt{2}}{2}, \frac{1 - \sqrt{2}}{2}$
23. $\frac{2 + \sqrt{3}}{4}, \frac{2 - \sqrt{3}}{4}$
24. $\frac{1 + 2i}{2}, \frac{1 - 2i}{2}$
25. $\frac{3 - i}{3}, \frac{3 + i}{3}$
26. $\frac{a - \sqrt{b}}{c}, \frac{a + \sqrt{b}}{c}$
27. $\frac{a - ib}{c}, \frac{a + ib}{c}$

EXERCISE 86. CHAPTER MASTERY-TEST

1. The function $3x^2 - 2x - 5$ is of the ________ degree.

2. (*a*) Draw the graph of the function $3x^2 - 2x - 5$.
(*b*) From the graph give the roots of the equation

$$3x^2 - 2x - 5 = 0.$$

3. Solve the equation $2x^2 - 5x + 3 = 0$ by factoring.

4. Solve the equation $x^2 - 4x - 5 = 0$ by completing the square.

5. Solve the equation $2x^2 - 3x - 4 = 0$ by the formula.

6. Without solving the equation $3x^2 - 4x - 2 = 0$:
(*a*) Tell the sum, the product, and kind of its roots;
(*b*) Does the graph of $3x^2 - 4x - 2$ cut the x-axis?

7. Form the equation whose roots are $1 + \sqrt{3}$ and $1 - \sqrt{3}$.

8. Tell the sum, the product, and the kind of roots of
(*a*) $2x^2 - 3x + 4 = 0$, without solving the equation.
(*b*) Does the graph of $2x^2 - 3x + 4$ cut the x-axis?

9. The graph of every second degree function of one variable is a ________.

10. (*a*) Tell the sum, the product, and the kind of roots of the equation $4x^2 - 12x + 9 = 0$.
(*b*) Does the graph of $4x^2 - 12x + 9$ cut the x-axis?

11. If a man travels 120 miles by auto and returns at a rate which is 10 miles an hour more, he will require 7 hours for the whole trip. What was his rate each way?

12. A boy was mowing a lawn, 40 ft. wide and 90 ft. long. How wide a border must he cut around its outside lines to have completed five sixths of his work?

13. $S = at + \frac{1}{2}gt^2$. Express t as a function of a, S, and g.

14. One root of $2x^2 - 15x + k = 0$ is twice the other. Find the value of k.

EXERCISE 87. CUMULATIVE REVIEW

1. Find the value of $\dfrac{2x^2 - 3x - 4}{2x^3 - 1}$ when $x = \frac{3}{2}$.

2. A boy is plowing a field which is 30 rods wide and 50 rods long. How wide a strip must he plow around the outside so that 2 acres (320 sq. rd.) shall remain to be plowed?

3. A man agreed to work at a certain job on condition that he receive \$9 for each day that he worked. His employer agreed on the condition that the man forfeit \$5 for each day that he failed to appear until the job was finished. At the end of 40 days (excluding Sundays) he received \$220. How many days did he work?

4. Rationalize the denominator and find the value to hundredths:

(*a*) of $\dfrac{1 + \sqrt{5}}{\sqrt{2}}$ (*b*) of $\dfrac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}}$

5. Write as a single fraction in lowest terms. Check, letting $a = 2$.

$$\frac{\dfrac{a}{a+2} - \dfrac{a}{a-3}}{\dfrac{a^2}{a+2} - \dfrac{a^2}{a-3}}$$

6. Solve the equation $x^2 - 2bx + c = 0$ by completing the square.

7. Find x to the nearest hundredth: $x^2 - .2x - 2.05 = 0$.

8. A and B working together can do a piece of work in 6 days; they can also do the work if A works 10 days and B works 3 days. How many days would it take each to do the work alone?

9. If a certain number be divided by the sum of its digits, the quotient is 5 and the remainder is 2. If the digits be reversed, the sum of the resulting number and 58 is twice the given number. Find the number.

10. Represent graphically the function $x^2 + 2x$. From this graph, determine the roots of the equation $x^2 + 2x = 5$ to the nearest tenth.

11. A fast train runs 8 miles per hour faster than a certain slower train. It requires 2 hours less time for a trip of 140 miles than does the slower train. Find the rate of each train.

12. A crew rowed 12 miles downstream, and back again in a total time of 8 hours. The rate of the current is known to be 2 miles per hour. At what rate did the crew row?

13. One machine can do a certain piece of work in 4 hours less time than another. When working together, they do the work in $1\frac{1}{2}$ hours time. In how many hours can each do it alone?

14. A farmer is plowing a field whose dimensions are 40 rods and 90 rods. How wide a border must he plow around the field in order to have completed $\frac{1}{2}$ his plowing?

15. Form the equation whose roots are $\frac{a - \sqrt{b}}{3}$ and $\frac{a + \sqrt{b}}{3}$.

16. Find the prime factors of:

(*a*) $x^{2a} - 12x^a + 32$ (*c*) $x^6 - 64a^6$

(*b*) $ax^4 - 16a$ (*d*) $x^4 - (3x - 2)^2$

17. Determine the roots of $x^2 + 4x - 5 = 0$ graphically.

18. Find the number of two digits such that, if the digits be reversed, the difference of the resulting number and the original number is 9, and their product is 736.

19. A rectangular field contains $2\frac{1}{4}$ acres. If its length be decreased by 10 rods, and its width by 2 rods, its area would be less by one acre. Find its length and width. (1A. = 160 sq. rd.)

20. The speed of an airplane is 90 miles an hour in calm weather. Flying with the wind, it can cover a certain distance in 4 hours but, when flying against the wind, it can cover only $\frac{3}{5}$ of this distance in the same time. What is the velocity of the wind?

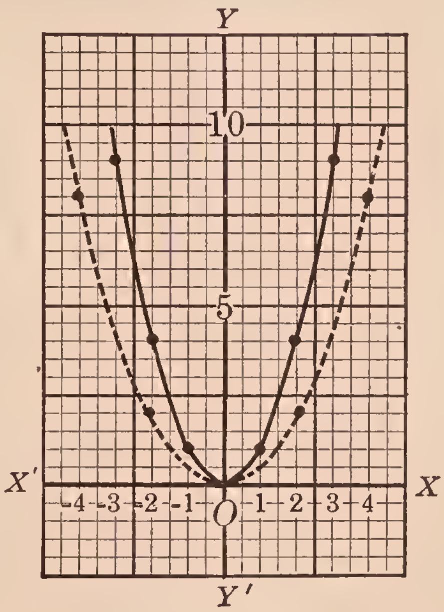
IX. GRAPHS OF EQUATIONS OF THE SECOND DEGREE

Two Variables

154. You should now learn **the names of certain curves** and their standard equations.

155. The **parabola.** Simplest form of its equation: $y = x^2$.

When x is	When y is
-3	$+9$
-2	$+4$
-1	$+1$
0	0
$+1$	$+1$
$+2$	$+4$
$+3$	$+9$



This graph is drawn on axes which have the same scale on both the x-axis and the y-axis.

Variations of this simple equation produce easily understood changes in this standard graph.

Example. Consider $y = \frac{1}{2}x^2$. Here the ordinate is $\frac{1}{2}$ *the square of the abscissa.* We can get the graph from that above by taking one-half of each ordinate, *without computing the new ordinates arithmetically.* The resulting graph is shown by a dotted curve. (Thus, above, when $x = 2$, $y = 4$. In this example, when $x = 2$, $y = \frac{1}{2}(4)$, or 2.)

We shall call this *sketching the graph.*

156. Many laws of nature and relations are pictured by this parabola or modifications of it. (Continued from p. 145.)

Example. $S = \frac{1}{2} gt^2$, or $S =$ (about) $16\, t^2$ represents the distance through which an object will fall in t seconds.

S changes as the square of t changes.

If this graph is drawn on the small scale shown on page 145, the parabola becomes very narrow, as each ordinate for this graph must be 16 times as long as the corresponding ordinates on page 145.

EXERCISE 88

1. On a large sheet of graph paper draw the graph of $y = x^2$ for values of x from -5 to $+5$. (*Suggestion.* If you use paper like that used for the graph on page 145, take one large space for the unit, both vertically and horizontally.)

2. On the same sheet as used for Example 1, sketch the graph of $y = \frac{1}{3} x^2$. (Use a dotted line. Do not *compute* ordinates. Rather make each ordinate $\frac{1}{3}$ as long as the corresponding ordinate in Example 1.)

3. On the same sheet *sketch* the graph of $y = 1.5\, x^2$. (Use a dash line. See Example 2.)

4. $A = \pi r^2$ is the area of a circle. Let $\pi = 3.1$. Then $A = 3.1\, r^2$. Draw this graph as follows:

(*a*) On a large sheet, first draw with a dotted line the graph of $A = r^2$.

(*b*) Then sketch the graph of $A = 3.1\, r^2$, remembering that each ordinate for it will be a little over 3 times the corresponding ordinate of $A = r^2$.

5. $S = \pi r^2$ is the formula for the surface of a sphere.

(*a*) What will be the shape of the graph of this equation?

(*b*) Tell how to sketch it, following a plan like that for Example 4.

6. Can you sketch the graph of $y = x^2 + 5$ without computing values of y?

157. The **circle.** Simple form of its equation: $x^2 + y^2 = r^2$, where r is the radius and the center is at the origin.

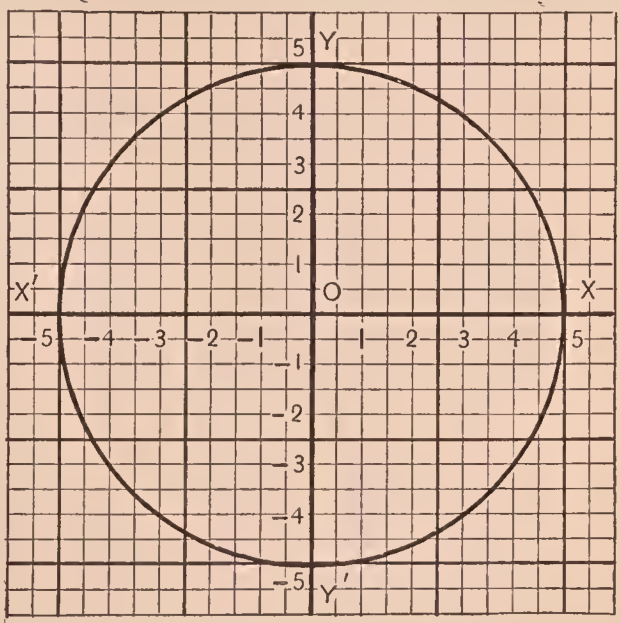
Example 1. Draw the graph of $x^2 + y^2 = 25$.

Solution. 1. $y^2 = 25 - x^2$, or $y = \pm\sqrt{25 - x^2}$.

2. When $x = -3$, $y = \pm\sqrt{25 - 9} = \pm\sqrt{16} = \pm 4$. Therefore when $x = -3$, $y = +4$; when $x = -3$, $y = -4$.

3. In this manner we obtain the table.

When $x = 0$	± 1	± 2	± 3	± 4	± 5
then $y = \pm 5$	± 4.8	± 4.5	± 4	± 3	0



This means: when $x = +1$, $y = +4.8$; when $x = -1$, $y = +4.8$; when $x = +1$, $y = -4.8$; when $x = -1$, $y = -4.8$, etc.

4. When x is greater than 5, y is *imaginary.*

Thus, when $x = 6$, $y = \pm\sqrt{25 - 36}$ or $\pm\sqrt{-11}$. This means that there are not any points on the graph for values of x greater than $+5$. Neither are there for values of x less than -5.

5. The graph appears at the right. The graph is a circle if the unit is of the same length on both the x-axis and the y-axis.

EXERCISE 89

1. What is the graph (or locus) of $x^2 + y^2 = 81$? Draw it without computing ordinates.

2. On the same set of axes sketch the graph of $x^2 + y^2 = 100$.

3. On the same set of axes sketch the graph of $x^2 + y^2 = 50$.

4. (*a*) Sketch the graph of $x^2 + y^2 = 36$ on axes which have units of the same length.

(*b*) Now sketch it on a set of axes, on which the unit for y is made one half as long as the unit for x.

158. The ellipse. Simple form of its equation: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Example 1. Draw the graph of $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1.$

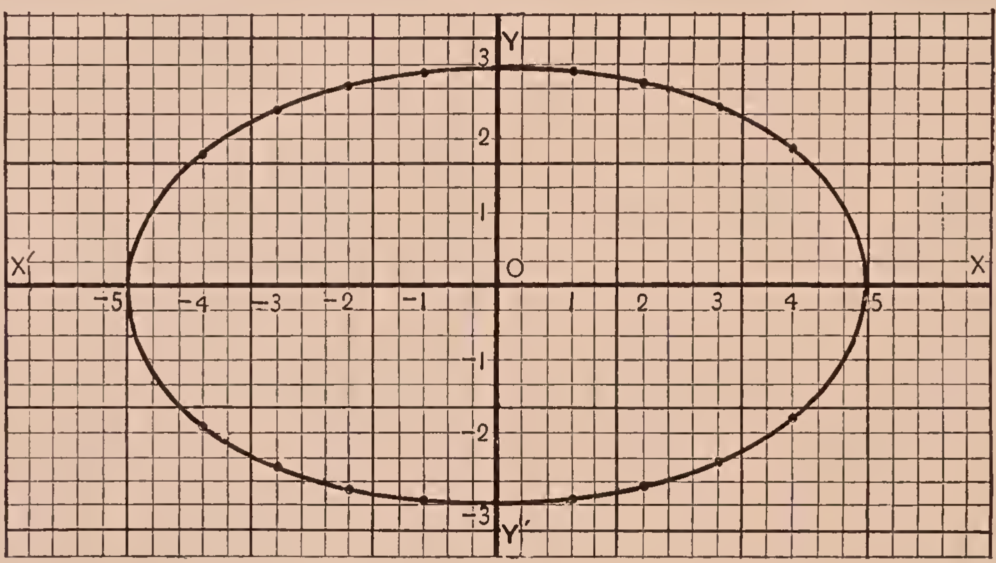
Solution. 1. Solving for y, you would get $y = \pm \frac{1}{5}\sqrt{225 - 9x^2}$.

2. If $x = 2$, $y = \pm 2.7$; so for each x, there are two values of y.

3. Also if $x = -2$, you would get $y = \pm 2.7$, since $(-2)^2 = 4$.

4. When x is more than 5, y is *imaginary;* also when x is less than -5.

When $x =$	0	± 1	± 2	± 3	± 4	± 5	over $+5$
Then $y =$	± 3	± 2.9	± 2.7	± 2.4	± 1.8	0	imaginary



5. The graph is an ellipse *provided the units on the y-axis and on the x-axis are of the same length.*

The greatest width in the x-direction is 10, just twice the square root of 25, the denominator of x^2. This is called the *major axis.* The greatest width in the y-direction is 6, just twice the square root of 9, the denominator of y^2. This is called the *minor axis.*

Example 2. What is the graph of $x^2 + 4y^2 = 64$?

Solution. 1. D_{64} $\dfrac{x^2}{64} + \dfrac{y^2}{16} = 1.$

2. The graph is an ellipse. Major axis = 16; minor axis = 8.

159. The equilateral hyperbola. The equation in simple form is $xy = k$, where k is an arithmetical number.

Example. Draw the graph of $xy = 6$.

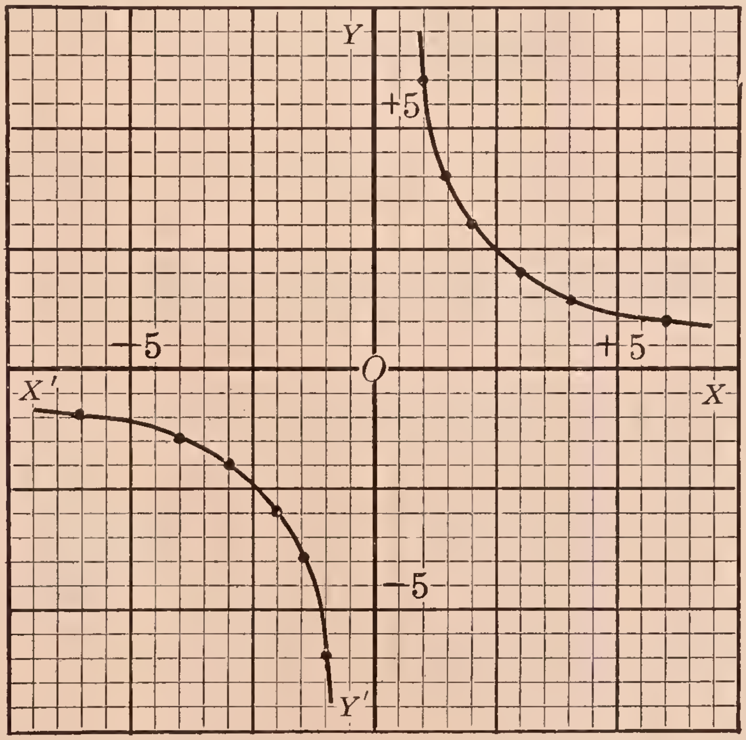
Solution. 1. $$y = \frac{6}{x}$$

2.

When $x = \frac{1}{2}$	1	$1\frac{1}{2}$	2	3	4	6	12	$-\frac{1}{2}$	-1	$-1\frac{1}{2}$	-2	-3	-4	-6	-12
then $y = 12$	6	4	3	2	$1\frac{1}{2}$	1	$\frac{1}{2}$	-12	-6	-4	-3	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$

3. The curve has the shape shown in the graph when the unit on the x-axis and y-axis is the same length.

Notice that y becomes smaller as x becomes larger; etc.



Example. You know the formula $d = rt$. Let $d = 100$. Then $rt = 100$. If the vertical axis is used for the values of r, and the horizontal axis for values of t, then the graph will be an equilateral hyperbola. (Negative values of r and t do not have any meaning.)

EXERCISE 90

1. Draw the graph of $A = hb$, when $A = 50$.
2. Draw the graph of $S = \frac{1}{2}\,Ch$, when $S = 50$.

160.* The general **hyperbola**. Simple form of the equation:

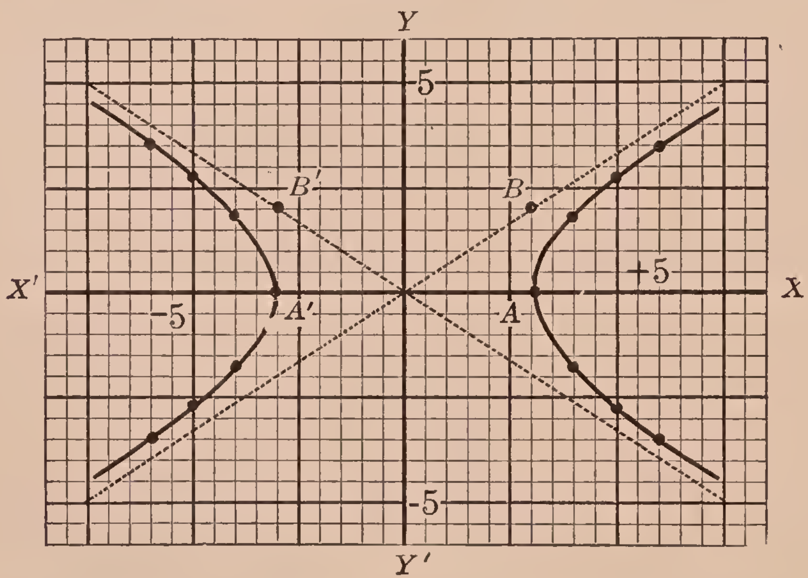
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Example. Draw the graph of $\frac{x^2}{9} - \frac{y^2}{4} = 1.$

Solution. 1. Solving for y, you would get $y = \pm \frac{2}{3}\sqrt{x^2 - 9}$.
2. If x is less than 3 or greater than -3, y is imaginary.
3. If $x = +3$, or -3, y is zero.
4. If $x = +4$, you get $y = \pm 1.8$.
Also, if $x = -4$, you get $y = \pm 1.8$.

5.

∴ When $x =$	0	± 1	± 3	± 4	± 5	± 6
then $y =$	imaginary	imaginary	0	$+1.8$	$+2.7$	$+3.5$



6. The graph is a *hyperbola*. Its "center" is at the origin. Observe that the hyperbola consists of two pieces; that each half *looks like* a parabola, which draws closer and closer to the dotted lines.

These dotted lines are easily located. Point B is the point (3, 2). Its abscissa is the square root of the denominator of x^2; its ordinate is the square root of the denominator of y^2. Similarly B' is the point $(-3, +2)$. OB and OB' are called the **asymptotes of the hyperbola.**

A is on the x-axis just below point B; A' is on the x-axis just below point B'.

* This graph is not required by the C. E. E. B.

X. SYSTEMS INVOLVING QUADRATICS

Two Variables

161. Systems consisting of one linear and one quadratic equation.

Example. Is there a right triangle with hypotenuse 5 inches and altitude 1 inch more than its base?

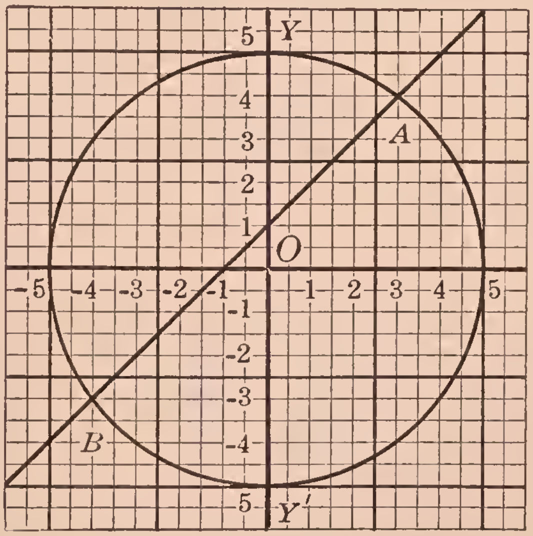
Graphical solution. 1. Let x = the base and y = the altitude.

2. $$\begin{cases} x^2 + y^2 = 25 & (1) \\ y - x = 1 & (2) \end{cases}$$

3. The graph of (1) is the circle with radius 5 inches.

4. For (2): when $x = 0$, $y = 1$;
   when $x = 2$, $y = 3$.

5. At A: $x = 3$, $y = 4$.
   At B: $x = -4$, $y = -3$.



Each is a solution of both equations.

(Only the values $x = 3$ and $y = 4$ have any meaning for this problem, of course.)

Algebraic solution. 1. From (2)
$$y = x + 1.$$

2. Substitute in (1).
$\therefore x^2 + x^2 + 2x + 1 = 25$,
or $2x^2 + 2x - 24 = 0$.

3. $\therefore x^2 + x - 12 = 0$. $\quad \therefore (x + 4)(x - 3) = 0$.

4. $\therefore x = -4$ | and $x = +3$
$\therefore y = -4 + 1$, or -3 | $\therefore y = 3 + 1$, or 4

Hence the common solutions are: $\begin{cases} x = 3, y = 4; \\ x = -4, y = -3. \end{cases}$

Check: For $x = -4$, $y = -3$
In (1): Does $16 + 9 = 25$? Yes
In (2): Does $(-3) - (-4) = 1$? Yes

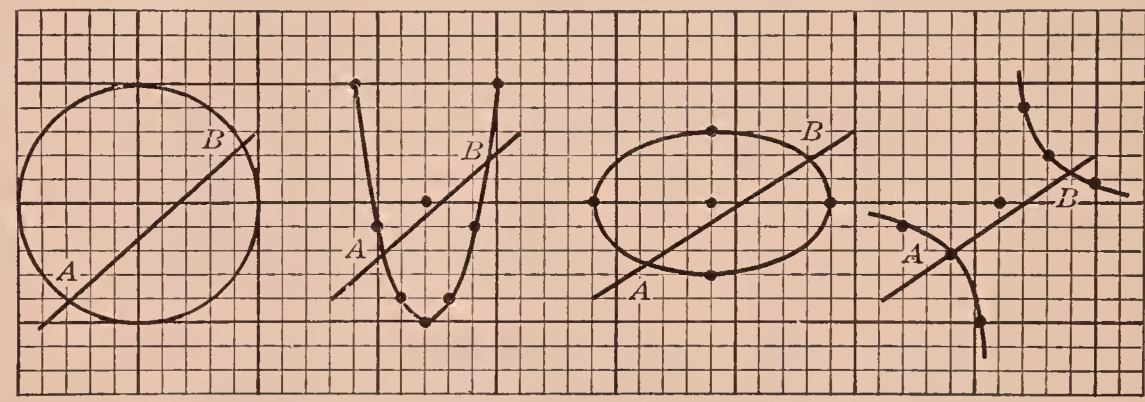
For $x = 3$, $y = 4$
In (1): Does $16 + 9 = 25$? Yes
In (2): Does $4 - 3 = 1$? Yes

162. A system consisting of one linear and one quadratic equation always has two solutions.

Geometric explanation. The graph of the linear equation is a straight line.

The graph of the quadratic equation is a circle, a parabola, an ellipse, or a hyperbola.

Intersection points therefore occur in figures such as:



The coördinates of *A* are a solution of the system; those of *B* are also a solution. Hence there are two solutions.

If the straight line does not cut the curve, the solutions are imaginary.

Example. Solve the system $\begin{cases} 3c + 2d = -2 & (1) \\ cd + 8c = 4 & (2) \end{cases}$

Solution. 1. From (1), $c = \dfrac{-2 - 2d}{3}$. (3)

2. Substituting in (2), $d\left(\dfrac{-2-2d}{3}\right) + 8\left(\dfrac{-2-2d}{3}\right) = 4.$ (4)

3. Simplifying, $-2d - 2d^2 - 16 - 16d - 12 = 0.$ (5)

4. $\therefore 2d^2 + 18d + 28 = 0$, or $d^2 + 9d + 14 = 0$.

5. Solving for d, $d = -2$, or $d = -7$.

6.	When $d = -2$, in (1), $3c - 4 = -2$ or $c = \frac{2}{3}$.	When $d = -7$, in (1), $3c - 14 = -2$ or $c = 4$.

7. $\therefore$ The solutions are $A.\ \begin{cases} c = \frac{2}{3} \\ d = -2 \end{cases}$ $B.\ \begin{cases} c = 4 \\ d = -7 \end{cases}$

These solutions check when substituted in (1) and (2).

EXERCISE 91

Solve the following systems:

1. $\begin{cases} x^2 + y^2 = 85 \\ x - y = 1 \end{cases}$

2. $\begin{cases} 4r^2 + t^2 = 25 \\ 2r + t = 7 \end{cases}$

3. $\begin{cases} xy = 28 \\ x + y = 11 \end{cases}$

4. $\begin{cases} a^2 + b^2 = 130 \\ a - b = 8 \end{cases}$

5. $\begin{cases} 2r + t = 7 \\ rt = 6 \end{cases}$

6. $\begin{cases} x^2 - xy - y^2 = 19 \\ x - y = 7 \end{cases}$

7. $\begin{cases} d - 2c = 8 \\ d^2 - 3cd = 22 \end{cases}$

8. $\begin{cases} xy = 18 \\ y - x = 3 \end{cases}$

9. $\begin{cases} x^2 - xy + y^2 = 63 \\ x - y = -3 \end{cases}$

10. $\begin{cases} x^2 + y^2 = 101 \\ x + y = 9 \end{cases}$

11. $\begin{cases} x^2 + y^2 + xy = 39 \\ x + y = -2 \end{cases}$

12. $\begin{cases} 2y + 2x = 5xy \\ 2x + 2y = 5 \end{cases}$

13. $\begin{cases} x^2 + y^2 = 52 \\ x - y = 2 \end{cases}$

14. $\begin{cases} 2x^2 + y^2 = 54 \\ x - \frac{1}{2}y = 0 \end{cases}$

15. $\begin{cases} x^2 - 3xy + y^2 = -11 \\ y - x = -1 \end{cases}$

16. $\begin{cases} y - 3x = 7 \\ x^2 + xy = 2 \end{cases}$

17. $\begin{cases} 4x^2 - xy = 2(x + y) \\ y - x = 1 \end{cases}$

18. $\begin{cases} x^2 + y^2 = a^2 \\ 2x - y = a \end{cases}$

19. $\begin{cases} y^2 - x^2 = 20 \\ x = \frac{2}{3}y \end{cases}$

20. $\begin{cases} x^2 + y^2 - 6y = 0 \\ y + 2x = 0 \end{cases}$

21. $\begin{cases} 4x^2 + y^2 = 40 \\ 2x + y = 8 \end{cases}$

22. $\begin{cases} x - y = 1 \\ xy = a^2 + a \end{cases}$

23. $\begin{cases} x^2 + y^2 = 25 \\ x - \frac{3}{4}y = 0 \end{cases}$

24. $\begin{cases} x + y = -3 \\ xy = -54 \end{cases}$

25. $\begin{cases} x^2 + y^2 = 16 - 6x \\ y = 2x + 1 \end{cases}$

26. $\begin{cases} x^2 - xy = -7 \\ 4x - y = 0 \end{cases}$

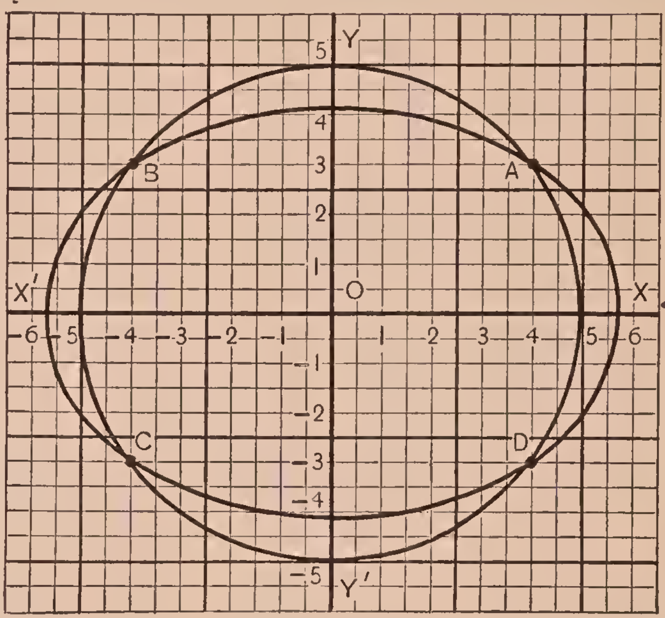
Solve Ex. 27 and 28 graphically and algebraically.

27. $\begin{cases} x^2 + y^2 = 4 \\ x + y = 4 \end{cases}$

28. $\begin{cases} y - x^2 = 0 \\ y - 4x + 4 = 0 \end{cases}$

163. Systems consisting of two quadratics. Read this and the following page as preparation for page 156.

Example 1. Solve the system
$$\begin{cases} x^2 + y^2 = 25 & (1) \\ x^2 + 2y^2 = 34 & (2) \end{cases}$$



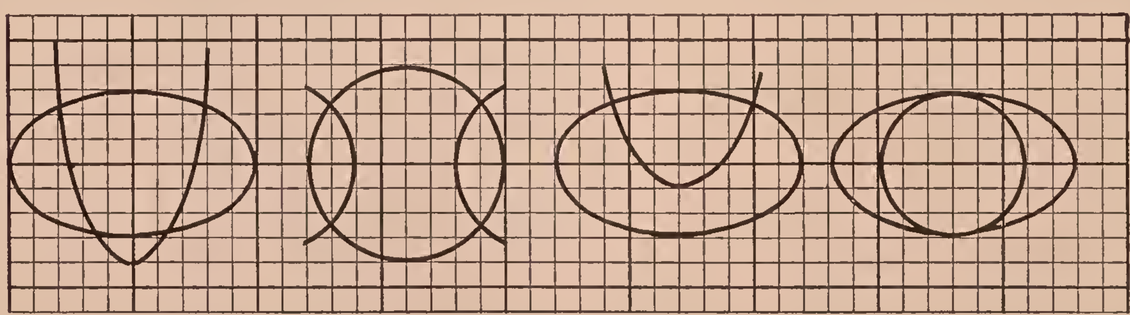
Graphical solution. 1. You know that the graph of (1) is a circle and of (2) is an ellipse.

2. You know that the coordinates of the intersection points should be common solutions.

At A: $x = +4,\ y = 3$
At B: $x = -4,\ y = 3$
At C: $x = -4,\ y = -3$
At D: $x = +4,\ y = -3$

These do satisfy both equations when substituted. There are *four solutions*, each consisting of *real* numbers.

As you know, the graph of a quadratic equation having two variables is a *circle*, an *ellipse*, a *parabola*, or a *hyperbola*. The graphical solution of a system of two such quadratics will produce figures like the following:



In no case are there more than four points of intersection, and hence four *real* common solutions. But these figures make it clear that there may be only two points of intersection, or even none at all.

Example 2. Solve the system $\begin{cases} x^2 + y^2 = 25 & (1) \\ x^2 - 3y + 3 = 0 & (2) \end{cases}$

Solution. 1. Subtracting, $y^2 + 3y - 28 = 0$

2. $\therefore\ (y + 7)(y - 4) = 0$

3. $\therefore\ y = -7$; or $y = +4$

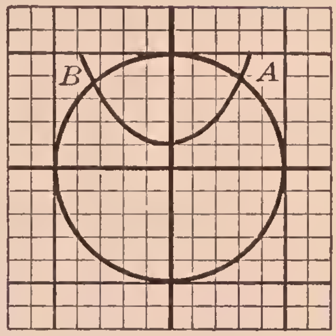
4. In (1): when $y = +4$:

$x^2 = 9$; or $x = \pm 3$.

5. In (1): when $y = -7$:

$x^2 = 25 - 49$, or $x^2 = -24$

$\therefore\ x = \pm\sqrt{-24}$, or $x = \pm 2i\sqrt{6}$.



There are four solutions:

$A: \begin{cases} x = +3 \\ y = +4 \end{cases}$ $\quad B: \begin{cases} x = -3 \\ y = +4 \end{cases}$

$C: \begin{cases} x = +2i\sqrt{6} \\ y = -7 \end{cases}$ $\quad D: \begin{cases} x = -2i\sqrt{3} \\ y = -7 \end{cases}$

NOTE. The circle ($x^2 + y^2 = 25$) cuts the parabola ($x^2 - 3y + 3 = 0$) at points A and B whose coördinates are solutions A and B.

This figure makes it appear that two imaginary solutions C and D correspond to two missing points of intersection.

Example 3. Solve the system $\begin{cases} x^2 + y^2 = 25 & (1) \\ x^2 - y = +5 & (2) \end{cases}$

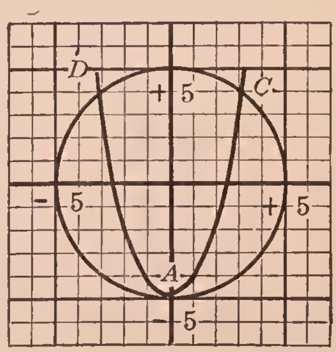
Solution. 1. From (2): $x^2 = y + 5$

2. $\therefore\ y^2 + y - 20 = 0$

3. $\therefore\ y = 4$; $y = -5$

In (1): when $y = -5$: $x^2 = 0$, or $x = (\pm)0$.

In (1): when $y = +4$: $x^2 = 9$, or $x = \pm 3$.



The solutions are:

$A: \begin{cases} x = 0 \\ y = -5 \end{cases}$ $\quad B: \begin{cases} x = 0 \\ y = -5 \end{cases}$

$C: \begin{cases} x = +3 \\ y = +4 \end{cases}$ $\quad D: \begin{cases} x = -3 \\ y = +4 \end{cases}$

In the figure, points C and D correspond to solutions C and D; and point A must correspond to solutions A and B.

Evidently, two identical solutions correspond to a point of tangency of the two graphs.

164. A system consisting of two quadratics having two variables always has four common solutions; two or four of these may be imaginary or complex numbers; two or four may be identical *real* solutions.

These solutions can be obtained algebraically for only certain systems. Description of the methods of solution is so lengthy that it is better to depend upon illustrative solutions.

One method is illustrated in Examples 2 and 3 on page 155.

Observe that one variable was eliminated, and then two values of the second variable were found. For each of these values of the second variable *two* values of the first variable were found, which, combined separately with the first variable, gave two solutions. Thus there resulted four solutions.

A second method is illustrated by the solution below.

Example. Solve the system $\begin{cases} a^2 + ab + b^2 = 63 & (1) \\ a^2 - b^2 = -27 & (2) \end{cases}$

Solution. 1. $M_3(1)$ $3a^2 + 3ab + 3b^2 = 189$ (3)

2. $M_7(2)$ $7a^2 - 7b^2 = -189$ (4)

3. (3) + (4) $10a^2 + 3ab - 4b^2 = 0$

4. $\therefore (5a + 4b)(2a - b) = 0. \therefore (5a + 4b) = 0$, and $(2a - b) = 0$.

5. Now form and solve the following two systems:

$\begin{cases} a^2 - b^2 = -27 \\ 2a - b = 0 \end{cases}$

You will find the solutions:

A: when $a = -3, b = -6$

B: when $a = +3, b = +6$

$\begin{cases} a^2 - b^2 = -27 \\ 5a + 4b = 0 \end{cases}$

You will find the solutions:

C: when $a = -4\sqrt{3}, b = +5\sqrt{3}$

D: when $a = +4\sqrt{3}, b = -5\sqrt{3}$

The plan in this solution is to combine the given equations so that a quadratic is secured which can be factored, and thus give two linear equations.

NOTE. Observe that the term not containing an unknown was eliminated. *This method is suggested only when the sum of the exponents in each term which contains an unknown is two.*

Thus: in a^2, the exponent is 2; in ab, the sum of the exponents of a and b is 2.

EXERCISE 92

Solve the following systems of equations:

1. $\begin{cases} 2x^2 - y^2 = -1 \\ x^2 + 2y^2 = 22 \end{cases}$

2. $\begin{cases} 2a^2 + b^2 = 7 \\ a^2 - 2b^2 = -4 \end{cases}$

3. $\begin{cases} 4x^2 - 3y^2 = -11 \\ 4x^2 + 4y^2 = 17 \end{cases}$

4. $\begin{cases} 2x^2 - 3y^2 = -6 \\ 4x^2 - y^2 = 8 \end{cases}$

5. $\begin{cases} 4x^2 + 7y^2 = 32 \\ 11y^2 - 3x^2 = 41 \end{cases}$

6. $\begin{cases} 2r^2 - 5s^2 = 13 \\ rs = -3 \end{cases}$

7. $\begin{cases} 2x^2 - 3y^2 = 19 \\ xy = -6 \end{cases}$

8. $\begin{cases} x^2 + y^2 = 40 \\ xy = 12 \end{cases}$

9. $\begin{cases} a^2 - b^2 = 3 \\ ab = -2 \end{cases}$

10. $\begin{cases} m^2 + n^2 = 10 \\ mn = -5 \end{cases}$

11. $\begin{cases} x^2 + 3xy = 28 \\ xy + 4y^2 = 8 \end{cases}$

12. $\begin{cases} x^2 - xy = 4 \\ x^2 + y^2 = 10 \end{cases}$

13. $\begin{cases} 3cd + d^2 = -14 \\ c^2 - cd = 30 \end{cases}$

14. $\begin{cases} 2x^2 - 3xy = 2 \\ 4x^2 + 9y^2 = 10 \end{cases}$

15. $\begin{cases} 4x^2 + 3y^2 = 28 \\ 2x^2 + xy + y^2 = 8 \end{cases}$

16. $\begin{cases} m^2 + mn = 8 \\ mn - n^2 = -12 \end{cases}$

17. $\begin{cases} x^2 + y^2 = 37 \\ xy = 6 \end{cases}$

18. $\begin{cases} x^2 - y^2 = 19 \\ 2x + y^2 = 5 \end{cases}$

19. $\begin{cases} 2x^2 - xy = 2 \\ 4x^2 + y^2 = 10 \end{cases}$

20. $\begin{cases} 3ab - 2b^2 = 0 \\ 2a^2 - ab = 2 \end{cases}$

21. $\begin{cases} n^2 + 3mn = 2 \\ 9m^2 + 2n^2 = 9 \end{cases}$

22. $\begin{cases} x^2 - 4y = 20 \\ x^2 + y^2 = 25 \end{cases}$

23. $\begin{cases} x^2 + xy = -6 \\ xy - y^2 = -35 \end{cases}$

24. $\begin{cases} 2x^2 - xy = 28 \\ x^2 + 2y^2 = 18 \end{cases}$

Solve the following systems for x and y:

25. $\begin{cases} x^2 + y^2 = 10a^2 \\ xy = 3a^2 \end{cases}$

26. $\begin{cases} x - y = -3a \\ 3x + 1y = 4a \end{cases}$

27. $\begin{cases} 3x^2 + y^2 = 2s^2 \\ x^2 - 4y^2 = 5s^2 \end{cases}$

28. $\begin{cases} d^2x^2 + c^2y^2 = c^2 + d^2 \\ xy = 1 \end{cases}$

165. Equivalent systems of equations are two different systems which have the same common solutions.

166. An equation is **rational** if the variables do not appear under a radical sign or with an exponent like $\frac{1}{2}$. (See § 111.)

An equation is **integral** if the variables do not appear in a denominator.

The **degree of a rational integral equation** is the sum of the exponents of the variables in that term in which said sum is largest; thus the degree of $x^2 - 2y^3 + xy = 0$ is 3.

Rule. Two rational and integral equations having two variables, whose degrees are *m* and *n* respectively, have *mn* common solutions provided the system cannot be reduced to an equivalent system consisting of equations of lower degree.

Thus, a system consisting of a cubic equation and a quadratic equation should have $3 \cdot 2$ or 6 common solutions. Some of these may be imaginary, and some may be repeated solutions.

167. Solution of systems which are reducible by division to equivalent systems whose equations are of lower degree.

Example. Solve the system
$$\begin{cases} x^4 - y^4 = 240 & (1) \\ x^2 + y^2 = 20 & (2) \end{cases}$$

Solution. 1. By the rule of § 166, it would appear that there should be $4 \cdot 2$ or 8 common solutions.

In this example, however, $x^4 - y^4$ can be divided by $x^2 + y^2$.

2. Dividing (1) by (2), $x^2 - y^2 = 12.$ (3)

3. Now form the system
$$\begin{cases} x^2 - y^2 = 12 & (3) \\ x^2 + y^2 = 20 & (2) \end{cases}$$

This system will have $2 \cdot 2$ or 4 common solutions.

4. (3) + (2): $2x^2 = 32;\quad x^2 = 16;\quad x = \pm 4.$

5. When $x = +4,\ 16 + y^2 = 20;\ y^2 = 4;\ y = \pm 2.$

$\therefore x = 4, y = 2$, and $x = 4, y = -2$ are two solutions.

6. When $x = -4,\ 16 + y^2 = 20;\ y^2 = 4;\ y = \pm 2.$

$\therefore x = -4, y = 2$, and $x = -4, y = -2$ are two more solutions.

Check all four solutions by substituting in equations (1) and (2).

EXERCISE 93

Solve the following systems; check the solutions by substituting in both equations:

1. $\begin{cases} x^2 - y^2 = 14 \\ x + y = 7 \end{cases}$

2. $\begin{cases} c^2 - d^2 = 56 \\ c - d = 4 \end{cases}$

3. $\begin{cases} x^4 - y^4 = 65 \\ x^2 + y^2 = 13 \end{cases}$

4. $\begin{cases} m^4 - n^4 = 240 \\ m^2 - n^2 = 12 \end{cases}$

5. $\begin{cases} x(x - y) = 50 \\ y(x - y) = 25 \end{cases}$

6. $\begin{cases} a^3 - 3a^2b = 54 \\ a - 3b = 6 \end{cases}$

7. $\begin{cases} x^2 + 2xy - 3y^2 = 17 \\ x + 3y = 17 \end{cases}$

8. $\begin{cases} c^3 - d^3 = 133 \\ c - d = 7 \end{cases}$

9. $\begin{cases} 2x^2 + 11xy + 12y^2 = 63 \\ 2x + 3y = 9 \end{cases}$

10. $\begin{cases} x^3 + y^3 = 35 \\ x + y = 5 \end{cases}$

11. $\begin{cases} x^3 - y^3 = 218 \\ x^2 + xy + y^2 = 109 \end{cases}$

12. $\begin{cases} x^3 + y^3 = -217 \\ x + y = -7 \end{cases}$

13. $\begin{cases} x^3 - x^2y = 54 \\ x - y = 6 \end{cases}$

14. $\begin{cases} x^3 - 8y^3 = 208 \\ x - 2y = 4 \end{cases}$

15. $\begin{cases} a^2 - b^2 = 4a + 6b - 8 \\ a - b = 2 \end{cases}$

16. $\begin{cases} m^3 - n^3 = -117 \\ m - n = -3 \end{cases}$

17. $\begin{cases} 3a + b = 2 \\ 27a^3 + b^3 = 98 \end{cases}$

18. $\begin{cases} x^2 + xy - 6y^2 = 21 \\ xy + 3y^2 = 84 \end{cases}$

19. $\begin{cases} x - y = 3 \\ x^2y - xy^2 = 30 \end{cases}$

20. $\begin{cases} a^3 + b^3 = 124 \\ a + b = 4 \end{cases}$

21. $\begin{cases} x^4 + xy^3 = 84 \\ x^3y + y^4 = 28 \end{cases}$

22. $\begin{cases} 2x - y = 3 \\ 2x^2y - xy^2 = 15 \end{cases}$

23. $\begin{cases} p(1 + .03r) = 420 \\ p(1 + .07r) = 480 \end{cases}$

24. $\begin{cases} x^3 - y^3 = 19a^3 \\ x^2 + xy + y^2 = 19a^2 \end{cases}$

25. $\begin{cases} x^3 + y^3 = 72 \\ x^2 - xy + y^2 = 12 \end{cases}$

26. $\begin{cases} 8x^3 - y^3 = 133m^3 \\ 2x - y = 7m \end{cases}$

27. $\begin{cases} 2r + 3s = -6 \\ 2r^2 + rs - 3s^2 = -42 \end{cases}$

28. $\begin{cases} 27x^3 - 8y^3 = 208t^3 \\ 3x - 2y = 4t \end{cases}$

EXERCISE 94

1. Find two numbers whose sum is -9 and the sum of whose squares is 101.

2. The sum of the squares of two numbers is 1274 and the larger is 5 times the smaller. Find the numbers.

3. The product of two numbers is 5 and the sum of their squares is 26. Find the numbers.

4. Find two numbers whose sum is 10 and whose product is 9.

5. Find two numbers whose difference is 5 and whose product is 6.

6. The product of two numbers is 48 and the sum of their squares is 100. Find the numbers.

7. The sum of two numbers is 8; the square of the first number increased by the product of the two numbers is equal to 49 diminished by the square of the second number. Find the numbers.

8. The square of the length of the diagonal of a rectangle is 40 and the area of the rectangle is 12 square feet. Find the dimensions of the rectangle.

9. Find two numbers such that the square of the first diminished by the product of the two numbers shall be 15, and such that the difference between their squares shall be 21.

10. Find two numbers such that their product increased by the first number shall be 8, and such that their product increased by twice the second number shall be 12.

11. The area of a certain triangle is 108 square feet. Its base exceeds its altitude by 6 feet. What are the base and altitude of the triangle? Also, solve this problem graphically.

12. If the larger of certain two numbers be multiplied by the difference of the two numbers, the result is 44. The product of the two numbers is 77. What are the numbers?

13. In 6 hours, two boys row 16 miles downstream and back again. Their rate upstream is twice the rate of the current. Find the rate at which they row in still water and the rate of the current.

14. The sum of the squares of the two digits of a number is 74. If 18 be added to the number, the digits of the sum are the digits of the original number in reverse order. Find the number.

15. Find the sides of a rectangle if the area of the rectangle increased by the shorter side is 15 and if the area increased by the longer side is 16.

16. The sum of the volumes of two cubes is 72; and an edge of one cube plus an edge of the other cube is 6. Find the length of the edges of each cube.

17. Find the number of two digits such that if the digits be reversed, the new number minus the original number is 18, and their product is 403.

18. Find two numbers the sum of whose squares increased by the product of the numbers is 28, and the difference of whose cubes is 56.

19. The area of a trapezoid is 36 square inches. Its altitude is 3 inches. Find the two bases of the trapezoid if the square of the length of the lower base is 192 more than the square of the length of the upper base.

20. The sum of a certain fraction and its reciprocal is $\frac{10}{3}$. If three times the numerator is decreased by twice the denominator, the result is equal to -12. Find the fraction.

21. In an isosceles triangle, the vertex angle is 60° more than the square of either base angle. Find the angles of the triangle.

22. The hypotenuse of a right triangle is 13 inches. Its altitude exceeds its base by 7 inches. What are the base and altitude? Solve this problem also graphically.

EXERCISE 95. CHAPTER MASTERY-TEST

1. Determine the common solutions of the system

$$\begin{cases} x^2 + xy + y^2 = 3 \\ x + y = 2. \end{cases}$$ Check the solutions.

2. Determine the common solutions of the system

$$\begin{cases} x^2 - y^2 = 12 \\ x^2 + y^2 = 24. \end{cases}$$ Properly pair off the values of x and y.

3. Determine the common solutions of $\begin{cases} x^2 - y^2 = -3 \\ xy = -2. \end{cases}$

4. Determine the common solutions of

$$\begin{cases} 2x^2 - 3xy = -4 \\ 4xy - 5y^2 = 3. \end{cases}$$

5. (*a*) Sketch the graphs for the system $\begin{cases} x^2 + y^2 = 36 \\ xy = 4. \end{cases}$

(*b*) How many real common solutions are there?

6. (*a*) Sketch the graphs for the system $\begin{cases} x^2 + y^2 = 16 \\ x + y = 3. \end{cases}$

(*b*) How many real common solutions are there?

7. (*a*) Sketch the graphs for the system $\begin{cases} x^2 + y^2 = 16 \\ x + y = 6. \end{cases}$

(*b*) How many and what kind of common solutions are there?

(*c*) Verify this statement by solving the system algebraically.

(*d*) Check the solutions by substituting in both equations.

8. The area of a rectangular field is 216 square rods and its perimeter is 60 rods. What is its length and width?

9. The difference of the rates of a passenger and a freight train is 10 miles an hour. The passenger train requires 1 hour more for a trip of 175 miles than the freight train requires for a trip of 100 miles. Find the rate of each.

10. How many and what possible kinds of solutions does a system of two quadratics have? Illustrate by free hand graphs.

EXERCISE 96. CUMULATIVE REVIEW

1. Find the prime factors of each of the following:

(*a*) $c^2 - 9m^2 + 6m - 1$ (*c*) $(m - n)^2 - 5(m - n) - 6$

(*b*) $a^6 - 7a^3 - 8$ (*d*) $x^{12} - 125y^3$

2. Simplify: $\dfrac{a^3 + b^3}{a - b + c}\left(1 - \dfrac{ab + c^2}{a^2 - ab + b^2}\right).$

3. Rationalize the denominator and simplify $\dfrac{2\sqrt{6} - 3}{\sqrt{3} - \sqrt{2}}.$

4. Draw the graph of the function $x^2 + 3x - 2$ and from it determine the roots of $x^2 + 3x - 2 = 0$ correct to tenths.

5. (*a*) Solve the formula $s = at + \frac{1}{2}gt^2$ for t.

(*b*) Using your results as new formulas, find t correct to hundredths when $g = 32$, $a = 1000$, and $s = 7500$.

6. A picture 15 inches wide and 20 inches long is to be surrounded by a frame whose area shall be $\frac{2}{3}$ that of the picture. What must be the width of the frame?

7. Solve the equation $ax^2 + 2bx + c = 0$ by completing the square.

8. By the formula $S = \dfrac{n}{2}\left\{2a + (n - 1)d\right\}$, determine n when $S = 25$, $a = 7$, and $d = -1$.

9. A crew can row 12 miles downstream and back again in $5\frac{1}{3}$ hours. If the rate of the stream is 3 miles per hour, find the rate of the crew in still water.

10. Solve the following system for x and y and correctly group your answers; also check the solution:

$$\begin{cases} x^2 - y^2 = 16 \\ y^2 - 14 = x \end{cases}$$

11. (*a*) Determine graphically the common solutions of $xy = 8$ and $x - y = 5$.

(*b*) Solve the system algebraically.

XI. FACTORS AND EQUATIONS OF HIGHER DEGREE

The Factor Theorem

168. The **factor theorem** aids in solving special equations of higher degree, such as $x^3 + x^2 - 2 = 0$.

Example 1. Is $x - 1$ a factor of $x^3 + x^2 - 2$? If it is, it is an exact divisor of $x^3 + x^2 - 2$.

The division at the right shows that $x - 1$ *is* a factor. The factor theorem enables us to determine this without dividing. Take the 1 of $x - 1$, and substitute it in $x^3 + x^2 - 2$. The result is 0. To one who knows the *factor theorem*, this is enough to tell that $x - 1$ is a factor of $x^3 + x^2 - 2$.

$$\begin{array}{r|l} & x^2 + 2x + 2 \\ \hline x - 1 & x^3 + x^2 + 0x - 2 \\ & x^3 - x^2 \\ \hline & \quad + 2x^2 + 0x \\ & \quad 2x^2 - 2x \\ \hline & \qquad 2x - 2 \\ & \qquad 2x - 2 \\ \hline \end{array}$$

Statement of the theorem. *If the value of a rational and integral polynomial in x is 0 when a is substituted for x, then $x - a$ is a factor of the polynomial.*

Proof: 1. Let $P(x)$, [read the polynomial in x] be the dividend; let $x - a$ be a divisor of it; Q the quotient; and R the remainder.

2. Then $P(x) = (x - a) \cdot Q + R$. [See § 19]

3. Let $x = a$ on both sides. Then $P(a) = (a - a) \cdot Q + R$, or $P(a) = R$.

Here $P(a)$ means the value of the polynomial when x is a; and by the theorem this is to be zero.

4. $\therefore 0 = R$.

5. Hence $x - a$ is a factor.

Example 2. Factor $x^3 + x^2 - 4x - 4$.

Solution. 1. Any factor must end in a divisor of 4. Is $x - 1$ a factor? Does $1^3 + 1^2 - 4 - 4 = 0$?

No, therefore $x - 1$ is not a factor.

2. Is $x + 1$ a factor? *Now* we must substitute -1 since $x - (-1) = x + 1$. Does $-1 + 1 + 4 - 4 = 0$? Yes.

Therefore $x + 1$ is a factor.

3. The remaining factor is found by division at the right.

$$\begin{array}{r|l} & x^2 - 4 \\ \hline x + 1 & x^3 + x^2 - 4x - 4 \\ & \underline{x^3 + x^2} \\ & \qquad\quad -4x - 4 \\ & \qquad\quad \underline{-4x - 4} \end{array}$$

4. $\therefore x^3 + x^2 - 4x - 4$
$= (x - 1)(x^2 - 4)$
$= (x - 1)(x + 2)(x - 2)$

NOTE. On the next two pages is a clever short method of division which *may be studied before doing the next exercise, or which may be omitted entirely.*

EXERCISE 97

Find the factors of:

1. $x^3 - 2x - 4$
2. $x^3 + x - 2$
3. $x^3 + x + 2$
4. $x^3 - 2x^2 - 9$
5. $x^3 + 2x + 12$
6. $x^3 - x - 6$
7. $3x^3 + 4x - 7$
8. $2x^3 + 3x + 5$
9. $2x^3 + x - 3$
10. $2x^3 - 5x - 6$
11. $x^3 - 4x^2 + x + 6$
12. $y^3 + 2y^2 - 9y - 18$
13. $m^3 - 5m^2 + 3m + 9$
14. $r^3 + 3r^2 - 4r - 12$
15. $s^3 - 5s^2 - 2s + 24$
16. $2x^3 + 6x^2 + 3x - 2$
17. $m^3 - m^2 - 3m + 2$
18. $x^3 - 19x - 30$
19. $y^3 - y^2 - 5y - 3$
20. $2z^3 - 3z^2 - 3z + 2$

169. In Step 3 of the proof on page 164 there is proof of another useful theorem.

The remainder theorem. *The value of a rational and integral polynomial in x when x equals a is the remainder obtained when the polynomial is divided by $x - a$.*

170. Synthetic division is a short form of division which *can be* used when the divisor is a binomial. It shortens the computation necessary when applying the factor theorem. It can be studied in advance of Exercise 97, or can be omitted altogether.

Example 1. Divide $5x^3 - 14x^2 - 10$ by $x - 3$.

Ordinary Method

$$
\begin{array}{r|l}
 & 5x^2 + x + 3 \\
\hline
x - 3 & 5x^3 - 14x^2 + 0x - 10 \\
 & \underline{5x^3 - 15x^2} \\
 & \qquad\quad x^2 + 0x \\
 & \qquad\quad \underline{x^2 - 3x} \\
 & \qquad\qquad\quad 3x - 10 \\
 & \qquad\qquad\quad \underline{3x - \ \ 9} \\
\text{Remainder:} & \qquad\qquad\qquad\ \ -1
\end{array}
$$

Synthetic Method

$$
\begin{array}{r|rrrr}
x + 3 & 5x^3 & -14x^2 & +0x & -10 \\
 & & +15 & +3 & +9 \\
\hline
 & 5 & +1 & +3 \| & -1
\end{array}
$$

Quotient: $5x^2 + x + 3$
Remainder: -1

Explanation of the Synthetic Method:

1. The divisor $x - 3$ is changed to $x + 3$. (This permits *addition* in Steps 3 to 5, instead of *subtraction* as in the usual method.)
2. $5x^3 \div x = 5x^2$. Write 5 in the third line.
3. $(+5) \cdot (+3) = +15$. Write it below -14; *add*, getting $+1$.
4. $(+1) \cdot (+3) = +3$. Write it below 0; *add*, getting $+3$.
5. $(+3) \cdot (+3) = +9$. Write it below -10; *add*, getting -1.
   5, $+1$, and $+3$ are the coefficients of the quotient.
   Hence, the quotient $= 5x^2 + x + 3$. Remainder $= -1$.
   These results agree with the ones secured by the ordinary method.

Example 2. Divide $7x^4 - 29x^2 - 3$ by $x + 2$.

Solution. Change $x + 2$ to $x - 2$.

$$
\begin{array}{r|rrrrr}
x - 2 & 7x^4 & +0x^3 & -29x^2 & +0x & -3 \\
 & & -14 & +28 & +2 & -4 \\
\hline
 & 7 & -14 & -1 & +2 \| & -7
\end{array}
$$

Quotient: $7x^3 - 14x^2 - x + 2$
Remainder: -7

Check.

$$
\begin{array}{rrrrr}
7x^3 & -14x^2 & -x & +2 & \\
 & & x & +2 & \\
\hline
7x^4 & -14x^3 & -x^2 & +2x & \\
 & +14x^3 & -28x^2 & -2x & +4 \\
\hline
7x^4 & & -29x^2 & & +4 \\
 & & & & -7 \\
\hline
7x^4 & & -29x^2 & & -3
\end{array}
$$

EXERCISE 98

Divide by synthetic division, and check:

1. $x^3 + 2x^2 - 2x + 5$ by $x - 1$
2. $y^3 - 4y^2 + y + 6$ by $y + 1$
3. $x^3 + 2x^2 - x - 2$ by $x - 2$
4. $x^3 - 3x^2 - x + 3$ by $x - 3$
5. $2x^3 - 3x^2 + x - 8$ by $x - 2$
6. $x^4 + x^2 - 6$ by $x + 2$
7. $3t^4 - 8t^3 - 27$ by $t - 3$
8. $x^4 + 2x^3 - 5x^2 - 12$ by $x + 2$
9. $5r^3 + 6r^2 - r - 12$ by $r - 2$
10. $3x^4 + 11x^3 - 4x^2 - x - 8$ by $x + 4$
11. Is $x - 1$ a factor of $x^3 + 3x^2 - 4$?

Suggestion. Divide. What should the remainder be?

12. Is $x + 2$ a factor of $x^3 + x^2 + 4$?

13. By § 169, if $x^3 + x^2 - x - 2$ be divided by $x - 1$ the remainder is the value of $x^3 + x^2 - x - 2$, when 1 is substituted for x. *Find this value* by finding the remainder by synthetic division. Check it by substituting 1 for x in $x^3 + x^2 - x - 2$.

As in Example 13, find:

14. The value of $x^3 + x^2 - x - 2$ when $x = +2$. (This means, divide by $x - 2$, and find the remainder.)

15. (*a*) The value of $2x^3 - x^2 + x - 4$ when $x = -1$.
(*b*) Also when $x = +1$, (*c*) when $x = +2$.

16. (*a*) The value of $x^3 - x - 6$ when $x = 2$.
(*b*) Tell two factors of $x^3 - x - 6$.

17. (*a*) Divide $x^5 - 1$ by $x - 1$.
(*b*) Is $x - 1$ a factor of $x^5 - 1$?

18. (*a*) Is $x - 1$ a factor of $x^4 - 1$?
(*b*) Is $x + 1$ a factor of $x^4 - 1$?

171. Graph of a third degree function of one variable.

Example. Draw the graph of $x^3 - 4x^2 - 2x + 8$.

Solution. 1. Let $y = x^3 - 4x^2 - 2x + 8$

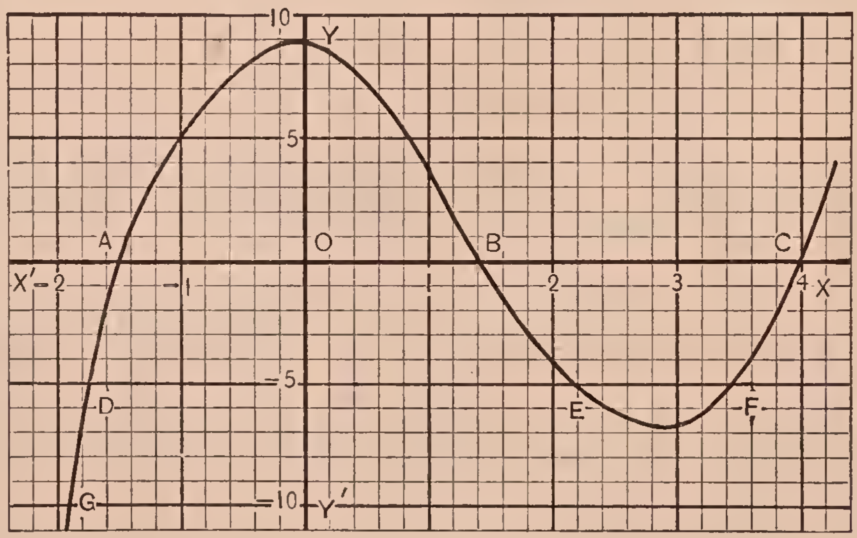
We must find values of y when $x = 0, 1, 2, 3, 4$, etc.

This can be done by direct substitution or as in § 170 as a consequence of the Remainder Theorem, § 169.

For example: when $x = 4$

$$\begin{array}{r|rrrr} & 1 & -4 & -2 & +8 \\ 4 & & +4 & 0 & -8 \\ \hline & 1 & 0 & -2 & \| \ 0 \end{array} \qquad \therefore y = 0 \text{ when } x = 4$$

When x =	0	1	2	3	4	5	− 1	− 2
then y =	3	8	− 4	− 7	0	23	5	− 12



The graph of every cubic function of one variable has this same general shape.

$y = 0$, where the curve crosses the x-axis. Hence

$x = -1.42$ (at A); $x = +1.42$ (at B); $x = +4$ (at C) are the roots of the equation $x^3 - 4x^2 - 2x + 8 = 0$.

The line $y = -5$ cuts the curve at D, E, and F.

At D, $x = -1.65$; at E, $x = 2.2$; at F, $x = 3.5$. Therefore -1.65, $+2.2$, and $+3.5$ are the roots of the equation $x^3 - 4x^2 - 2x + 8 = -5$. The line $y = -8$ cuts the curve only at G, where $x = -1.8$.

Hence $x = -1.8$ is the only *real root* of $x^3 - 4x^2 - 2x + 8 = -8$.

The other two roots are imaginary.

172. Algebraic solution of equations of higher degree is difficult except for special equations.

It is proved in higher mathematics that *every equation of degree n has n roots.*

Example 1. Solve the equation $x^4 + 4x^2 - 5 = 0$

Solution. 1. Factoring: $(x^2 - 1)(x^2 + 5) = 0$

2. $\therefore x^2 - 1 = 0;\quad x^2 = 1;\quad x = \pm 1$

or $x^2 + 5 = 0;\quad x^2 = -5;\quad x = +\sqrt{-5};\quad x = -\sqrt{-5}$

The four roots are: $+1;\ -1;\ \pm\sqrt{5};\ \pm i\sqrt{5}.$

Example 2. Solve the equation $x^3 + x - 2 = 0$

Solution. 1. By the factor theorem $(x - 1)(x^2 + x + 2) = 0.$

2. $\therefore x - 1 = 0;$ or $x^2 + x + 2 = 0$

3. $$\therefore x = 1;\ \text{or } x = \frac{-1 \pm \sqrt{1 - 8}}{2}$$

4. Hence $x = 1;\quad x = \dfrac{-1 + i\sqrt{7}}{2};\quad x = \dfrac{-1 - i\sqrt{7}}{2}$

EXERCISE 99

Solve the following equations:

1. $x^4 - 5x^2 + 4 = 0$
2. $c^4 - 26c^2 + 25 = 0$
3. $y^4 - 29y^2 + 100 = 0$
4. $x^4 - 8x^2 - 9 = 0$
5. $y^4 - 5y^2 - 36 = 0$
6. $t^4 + 2t^2 - 8 = 0$
7. $4x^4 - 4x^2 + 1 = 0$
8. $2y^4 - 9y^2 + 4 = 0$
9. $4m^4 - 17m^2 + 4 = 0$
10. $6x^4 - 5x^2 + 1 = 0$
11. $x^3 - x^2 - 5x + 5 = 0$
12. $x^3 - 2x^2 - x + 2 = 0$
13. $y^3 - 2y^2 - 6y - 3 = 0$
14. $x^3 + 2x^2 - x - 2 = 0$
15. $x^3 - 2x^2 + 1 = 0$
16. $x^3 - 2x + 1 = 0$
17. $x^3 - 3x^2 + 2 = 0$
18. $y^3 + 3y^2 - 2 = 0$
19. $y^3 - 4y^2 - 7y + 10 = 0$
20. $y^3 + 3y^2 - 4 = 0$
21. $2x^3 + x^2 - 6x + 3 = 0$
22. $z^3 - z^2 - 8z + 8 = 0$
23. $x^4 - x^3 - 8x^2 + 12x = 0$
24. $c^3 - 6c^2 + 11c - 6 = 0$
25. $x^3 + x^2 - 6x = 0$
26. $2x^3 - 5x^2 + x + 2 = 0$

XII. EXPONENTS AND RADICALS

173. We shall first prove the **five laws of exponents** for **positive integral exponents.** So far you have used these laws without having proved them.

174. Definition. When m is a positive integer,

$$x^m = x \cdot x \cdot x \cdots\cdots x. \qquad (m \text{ factors})$$

[Notice how this is written. The dots are read "and so on." The number of factors is indicated at the right or below.]

175. The multiplication law. $x^m \cdot x^n = x^{m+n}$

Proof. 1. $x^m = x \cdot x \cdot x \cdots\cdots x \qquad (m \text{ factors})$

2. $x^n = x \cdot x \cdot x \cdots\cdots x \qquad (n \text{ factors})$

3. $\therefore x^m \cdot x^n = \underbrace{\{x \cdot x \cdot x \cdots\cdots x\}}_{(m \text{ factors})} \cdot \underbrace{\{x \cdot x \cdot x \cdots\cdots x\}}_{(n \text{ factors})}$

4. $= x \cdot x \cdot x \cdots\cdots x \cdot x \cdot x \cdot x \cdots\cdots x \qquad (m + n \text{ factors})$

5. $\therefore x^m \cdot x^n = x^{m+n}$

176. The division law. $x^m \div x^n = x^{m-n}$ (m greater than n)

Proof. 1. $\dfrac{x^m}{x^n} = \dfrac{\not{x} \cdot \not{x} \cdot \not{x} \cdots\cdots \not{x} \cdot x \cdot x \cdots\cdots x}{\not{x} \cdot \not{x} \cdot \not{x} \cdots\cdots \not{x}} \qquad \begin{matrix}(m \text{ factors})\\(n \text{ factors})\end{matrix}$

2. $\therefore \dfrac{x^m}{x^n} = x \cdot x \cdots\cdots x. \qquad (m - n \text{ factors})$

3. $\therefore \dfrac{x^m}{x^n} = x^{m-n}$

177. The power of a power law. $(x^m)^n = x^{mn}$

Proof. 1. $(x^m)^n = x^m \cdot x^m \cdot x^m \cdot x^m \cdots\cdots x^m \qquad (n \text{ factors})$

2. $\therefore (x^m)^n = x^{m+m+m+\cdots\cdots+m} \qquad (n \text{ addends in exponent})$

3. $\therefore (x^m)^n = x^{mn}$

178. Power of a product law. $(xy)^n = x^n y^n.$

Proof. 1. $(xy)^n = (xy)(xy)(xy) \cdots\cdots (xy)$ $(n$ factors$)$

2. $\therefore (xy)^n = \underbrace{(x \cdot x \cdot x \cdots\cdots x)}_{(n \text{ factors})} \underbrace{(y \cdot y \cdot y \cdots\cdots y)}_{(n \text{ factors})}$ § 13, *b*

3. $\therefore (xy)^n = x^n y^n$

179. Power of a quotient law. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

Proof. 1. $\left(\frac{x}{y}\right)^n = \left(\frac{x}{y}\right)\left(\frac{x}{y}\right)\left(\frac{x}{y}\right) \cdots\cdots \left(\frac{x}{y}\right)$ $(n$ factors$)$

2. $\therefore \left(\frac{x}{y}\right)^n = \frac{x \cdot x \cdot x \cdots\cdots x \ (n \text{ factors})}{y \cdot y \cdot y \cdots\cdots y \ (n \text{ factors})}$

3. $\therefore \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

EXERCISE 100

In the following examples, *the literal exponents represent positive integers.* Find the results of the indicated operations:

1. $x^{10} \cdot x^4$	**12.** $z^{12} \div z^n$	**23.** $(z^4)^3$	**34.** $(-4a^3b)^3$
2. $m^6 \cdot m^8$	**13.** $t^{2n} \div t^3$	**24.** $(w^6)^3$	**35.** $(-\frac{1}{2}a)^3$
3. $y^4 \cdot y^8$	**14.** $x^{3m} \div x^m$	**25.** $(x^5)^6$	**36.** $(-\frac{2}{3}cd^2)^3$
4. $x^a \cdot x^b$	**15.** $y^{m+4} \div y^3$	**26.** $(y^3)^7$	**37.** $(\frac{3}{4}xy^2)^3$
5. $x^{2n} \cdot x^{5n}$	**16.** $z^m \div z^{m-1}$	**27.** $(m^4)^9$	**38.** $(x^2y^2w)^4$
6. $x^3 \cdot x^r$	**17.** $k^{r+2} \div k^r$	**28.** $(-a^2b)^2$	**39.** $(-a^2b^3)^5$
7. $y^{c-2} \cdot y^3$	**18.** $w^{3r+1} \div w^2$	**29.** $(-2ab)^3$	**40.** $(a^3)^n$
8. $z^{2n} \cdot z^5$	**19.** $x^{n+3} \div x^2$	**30.** $(2xy)^3$	**41.** $(b^m)^4$
9. $w^{n-1} \cdot w^n$	**20.** $x^{n-3} \div x^2$	**31.** $(3xy^2z)^3$	**42.** $(x^2y^3)^n$
10. $x^{3m} \cdot x^{m+1}$	**21.** $(x^3)^2$	**32.** $(4mn^2)^3$	**43.** $(r^4s^3)^n$
11. $y^{10} \div y^3$	**22.** $(y^2)^3$	**33.** $(-2x^2y)^3$	**44.** $(x^my^n)^t$
45. $\left(\frac{x^2}{y^3}\right)^2$	**46.** $\left(\frac{2a}{b}\right)^3$	**47.** $\left(\frac{3c^2}{5d}\right)^3$	**48.** $\left(-\frac{2m^3}{3n^2}\right)^3$

180. Roots of numbers:

(*a*) You know that a number like 9 has two square roots; that + 3 is its *principal* square root; and that the principal square root of 9 is indicated by $\sqrt{9}$ or by $9^{\frac{1}{2}}$. (See p. 100.)

(*b*) Similarly a number has *three cube roots, four fourth roots,* and *n nth roots.*

Thus, + 2 is a cube root of + 8, since $(+2)^3 = +8$.

The other two cube roots of + 8 are complex numbers. + 2 is *the principal cube root* of + 8.

− 2 is a cube root of − 8; since $(-2)^3 = -8$.

The other two cube roots of − 8 are complex numbers.

− 2 is *the principal cube root* of − 8.

Definition. *The principal nth root of a number x is indicated by* $\sqrt[n]{x}$. *n* is the **index** of the root; $\sqrt{\ }$ is the **radical sign**; *x* is the **radicand.**

The **principal root of a positive number** is positive.

Thus $\sqrt[3]{125} = +5$; $\sqrt[4]{16} = +2$.

The **principal odd root of a negative number** is negative.

Thus $\sqrt[3]{-8} = -2$; $\sqrt[5]{-32} = -2$; $\sqrt[3]{-27x^6} = -3x^2$.

181. Roots of numbers are also indicated by fractional exponents.

(*a*) *Why the exponent* $\frac{1}{2}$ *is used to indicate the square root.*

If $x^{\frac{1}{2}}$ obeys the multiplication law of exponents, then $(x^{\frac{1}{2}})(x^{\frac{1}{2}})$ should $= x^1$, or x. Then $x^{\frac{1}{2}}$ *ought to mean* $\sqrt{x}$, or $-\sqrt{x}$.

We define $x^{\frac{1}{2}}$ as being the same as $\sqrt{x}$.

(*b*) Similarly, if $(x^{\frac{1}{3}})(x^{\frac{1}{3}})(x^{\frac{1}{3}}) = x^1$, or x, then $x^{\frac{1}{3}}$ ought to be a cube root of x.

We define $x^{\frac{1}{3}}$ as being the *principal cube root* of x. ($\sqrt[3]{x}$).

(*c*) In general, **we define** $x^{\frac{m}{n}}$ **as being** $\sqrt[n]{x^m}$.

Thus $4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8$; $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$.

EXERCISE 101

Give the principal root indicated below:

1. $27^{\frac{1}{3}}$	7. $\sqrt[4]{16}$	13. $(-a^3x^3)^{\frac{1}{3}}$	19. $(\frac{1}{16}x^4)^{\frac{1}{4}}$
2. $16^{\frac{1}{2}}$	8. $(144)^{\frac{1}{2}}$	14. $\sqrt[4]{x^{12}}$	20. $\sqrt[6]{x^{12}}$
3. $64^{\frac{1}{3}}$	9. $81^{\frac{1}{4}}$	15. $(x^4y^4)^{\frac{1}{2}}$	21. $(64)^{\frac{1}{6}}$
4. $\sqrt[3]{-\frac{1}{8}}$	10. $(-125)^{\frac{1}{3}}$	16. $\sqrt[3]{\frac{1}{27}x^3}$	22. $\sqrt{.25}$
5. $100^{\frac{1}{2}}$	11. $\sqrt[5]{a^5b^5}$	17. $(\frac{1}{8}a^3)^{\frac{1}{3}}$	23. $(.36)^{\frac{1}{2}}$
6. $(-27)^{\frac{1}{3}}$	12. $32^{\frac{1}{5}}$	18. $\sqrt[4]{16a^4}$	24. $(.008)^{\frac{1}{3}}$
25. $\sqrt[3]{\dfrac{x^6}{y^6}}$	28. $\sqrt{\dfrac{x^4}{y^4}}$	31. $\sqrt{\dfrac{x^{2a}}{4}}$	34. $\left(\dfrac{-x^{3r}}{8}\right)^{\frac{1}{3}}$
26. $\left(\dfrac{-8}{b^3}\right)^{\frac{1}{3}}$	29. $\left(\dfrac{16}{a^4}\right)^{\frac{1}{4}}$	32. $\left(\dfrac{y^{2b}}{9}\right)^{\frac{1}{2}}$	35. $\left(\dfrac{-27a^3}{b^3}\right)^{\frac{1}{3}}$
27. $\sqrt[3]{\dfrac{x^{3a}}{27}}$	30. $\left(\dfrac{-32}{m^5}\right)^{\frac{1}{5}}$	33. $\sqrt{\dfrac{x^{2m}}{y^{2n}}}$	36. $\sqrt[4]{\dfrac{x^{4a}}{y^{4b}}}$

182. By definition $x^{\frac{m}{n}}$ means $\sqrt[n]{x^m}$. It can be proved that it also means $(\sqrt[n]{x})^m$.

For example: to find $4^{\frac{3}{2}}$, find $(\sqrt{4})^3$, or 2^3, or 8.

EXERCISE 102

Simplify:

1. $9^{\frac{3}{2}}$	4. $27^{\frac{2}{3}}$	7. $(-8)^{\frac{2}{3}}$	10. $(-32)^{\frac{3}{5}}$
2. $16^{\frac{3}{2}}$	5. $4^{\frac{5}{2}}$	8. $(-27)^{\frac{2}{3}}$	11. $(-8)^{\frac{4}{3}}$
3. $8^{\frac{2}{3}}$	6. 8_3	9. $(-32)^{\frac{2}{5}}$	12. $(-27)^{\frac{5}{3}}$

183. (Optional.) **What are the cube roots of 1?**

1. Let $x = \sqrt[3]{1}$, or $x^3 = 1$; or $x^3 - 1 = 0$
2. $\therefore (x-1)(x^2+x+1) = 0$
3. Completing the solution of this equation:

$$x_1 = 1;\ x_2 = \tfrac{1}{2}(-1 + i\sqrt{3});\ x_3 = \tfrac{1}{2}(-1 - i\sqrt{3}).$$

NOTE. The interesting fact is that the cube roots of 8 are:

$$2;\ 2\left[\tfrac{1}{2}(-1 + i\sqrt{3})\right];\ 2\left[\tfrac{1}{2}(-1 - i\sqrt{3})\right].$$

184. What meaning has the exponent zero?

If x^0 obeys the multiplication law of exponents then

$$x^n \cdot x^0 = x^{n+0}, \text{ or } x^n. \quad \therefore\ x^0 = x^n \div x^n, \text{ or } x^0 = 1.$$

This suggests the *following definition.*

Definition. $x^0 = 1$, except when $x = 0$.

Thus $5^0 = 1;\quad 5000^0 = 1;\quad x^0 = 1;\quad (-15)^0 = 1.$

185. What meaning has a negative exponent?

If x^{-3} obeys the multiplication law of exponents, then

$$x^{-3} \cdot x^{+3} = x^{-3+3} = x^0 = 1. \quad \text{Hence } x^{-3} = \frac{1}{x^3}.$$

This suggests the *following definition.*

Definition. $$x^{-n} = \frac{1}{x^n}$$

Thus: $x^{-4} = \frac{1}{x^4};\quad 2^{-3} = \frac{1}{2^3} = \frac{1}{8};\quad 4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{2}$

EXERCISE 103

Find the numerical value of, or the simplest form of:

1. 4^0	11. 3^{-3}	21. $8^{-\frac{1}{3}}$	31. $18^0 \cdot 2^{-4}$
2. y^0	12. 2^{-5}	22. $16^{-\frac{1}{2}}$	32. $3^{-2} \cdot 27$
3. $3x^0$	13. 6^{-2}	23. $27^{-\frac{1}{3}}$	33. $1000^0 \cdot 2^{-3}$
4. $(2x)^0$	14. $(-3)^{-1}$	24. $(-8)^{-\frac{1}{3}}$	34. 10^{-3}
5. $2^0 x$	15. $(-3)^{-2}$	25. $25^{-\frac{1}{2}}$	35. $4^{-\frac{3}{2}}$
6. 4^{-2}	16. $(-2)^{-3}$	26. $6^0 \cdot 3^{-2}$	36. 32×10^{-1}
7. 3^{-1}	17. $(+2)^{-3}$	27. $64 \cdot 2^{-3}$	37. 32×10^{-3}
8. 3^{-2}	18. $(-3)^{+2}$	28. $9^0 \cdot 9^{-\frac{1}{2}}$	38. 56×10^{-4}
9. 5^{-2}	19. $(-2)^{-4}$	29. $(64)^{-\frac{1}{3}}$	39. 5×10^{-5}
10. 8^0	20. $(-4)^{-2}$	30. $(-27)^{-\frac{1}{3}}$	40. 1×10^{-3}

Rewrite using positive exponents:

41. a^3b^4	42. $2r^{-4}$	43. $2a^{-2}b^4$	44. $10^{-2}x^5$

186. Negative exponents in fractions.

Example 1. $\dfrac{3a^{-3}}{4c^{-2}} = \dfrac{3}{a^3} \div \dfrac{4}{c^2} = \dfrac{3}{a^3} \cdot \dfrac{c^2}{4} = \dfrac{3c^2}{4a^3}$

Observe that a^{-3} of the numerator becomes a^{+3} in the denominator; that c^{-2} of the denominator becomes c^{+2} in the numerator.

Rule. Any factor of one term of a fraction may be transferred to the other term provided the sign of its exponent is changed.

Example 2. $\dfrac{5x^2y^{-4}z^3}{w^{-2}t} = \dfrac{5x^2w^2z^3}{y^4t}.$

Example 3. $\dfrac{3a^2b}{cd^3} = 3a^2bc^{-1}d^{-3}.$

EXERCISE 104

Rewrite with only positive exponents:

1. $\dfrac{x^{-3}y}{z^2}$
2. $\dfrac{3m}{n^{-2}}$
3. $\dfrac{4a^{-2}}{b^3}$
4. $\dfrac{5x^{-3}}{y^{-2}}$
5. $\dfrac{mn}{10^{-5}}$
6. $\dfrac{x^2}{(2y)^{-2}}$
7. $\dfrac{5m^2n^{-4}}{6r^2}$
8. $\dfrac{4a^{-3}b^2}{5c^2d^{-3}}$
9. $\dfrac{8a^{\frac{1}{2}}b^{-\frac{1}{3}}}{c^{\frac{1}{3}}}$
10. $\dfrac{k}{10^{-15}}$
11. $\dfrac{10^{-7}x^2}{y^3}$
12. $\dfrac{4 \cdot 3^{-1}\pi}{r^{-3}}$

Write without any denominator:

13. $\dfrac{5x^6}{y^3}$
14. $\dfrac{3c^4}{d^{-2}}$
15. $\dfrac{2m^3n^2}{r^5}$
16. $\dfrac{x^3y^{-2}}{z^4}$
17. $\dfrac{7cd}{m^4}$
18. $\dfrac{8x^{\frac{1}{2}}}{y^{\frac{2}{3}}}$
19. $\dfrac{5a^3b^{-2}}{2^{-1}c^{-2}}$
20. $\dfrac{6a^{\frac{2}{3}}}{b^{-\frac{3}{4}}}$
21. $\dfrac{1}{(1+i)^n}$
22. $\dfrac{1}{\left(1+\dfrac{j}{m}\right)^{mn}}$
23. $\dfrac{(1+i)^n - 1}{i}$
24. $\dfrac{1}{1.02^5} + \dfrac{1}{1.02^4} + \dfrac{1}{1.02^3} + \dfrac{1}{1.02^2} + \dfrac{1}{1.02}$

[Examples 21–24 are from the mathematics of investment.]

187. Meanings have been given to fractional, negative, and zero exponents so that each will obey the multiplication law of exponents. *It can be proved* that, with these meanings, they obey all the laws of exponents given in § 175 to § 179, on pages 170 to 171. The following exercise gives a little practice in use of these laws with these new exponents. Examples 30 to 50 are preparation for the study of the next chapter.

EXERCISE 105

Perform the indicated operation:

1. $r^{-3} \cdot r^4$

2. $s^{-2} \cdot s^3$

3. $x^{-4} \cdot xy^2$

4. $x^{\frac{1}{2}} \cdot x^{\frac{1}{3}}$

5. $y^{\frac{1}{4}} \cdot y^{\frac{3}{4}}$

6. $x^{\frac{2}{3}} \cdot x^{\frac{1}{6}}$

7. $m^{-\frac{1}{2}} \cdot m$

8. $n^{-\frac{1}{3}} \cdot n^{\frac{1}{2}}$

9. $n^{-\frac{3}{4}} \cdot n^{-\frac{1}{4}}$

10. $x^{-2} \cdot x^{-\frac{1}{2}}$

11. $t^2 \div t^{-1}$

12. $t^{-3} \div t^{-2}$

13. $x^{-2} \div x^{-3}$

14. $x^{-3} \div x^0$

15. $x \div x^{\frac{1}{2}}$

16. $x^2 \div x^{\frac{1}{4}}$

17. $x^{\frac{1}{2}} \div x^{\frac{1}{3}}$

18. $y^{\frac{2}{3}} \div y^{\frac{1}{3}}$

19. $x^{-\frac{3}{4}} \div x^{-\frac{1}{4}}$

20. $x^{-\frac{2}{3}} \div x^{-\frac{1}{3}}$

21. $(x^3)^{-2}$

22. $(x^{\frac{1}{2}})^2$

23. $(x^{-\frac{1}{3}})^2$

24. $(x^{\frac{1}{2}})^{-3}$

25. $(x^{\frac{1}{2}}y^{\frac{1}{3}})^2$

26. $(x^{-\frac{1}{2}}y^{\frac{1}{2}})^2$

27. $(x^{\frac{1}{4}} \cdot y^{-\frac{1}{2}})^4$

28. $(x^{-\frac{2}{3}})^{-\frac{1}{2}}$

29. $(x^{-\frac{3}{4}}y^{-\frac{1}{4}})^4$

30. $(x^{-\frac{3}{2}})^{\frac{4}{3}}$

31. $10^3 \times 10^2$

32. $10^4 \div 10^3$

33. $10^2 \div 10^3$

34. $10^{-2} \times 10^{-1}$

35. $10^{-3} \div 10^{-2}$

36. $10^{2.5} \times 10^{1.5}$

37. $10^{.25} \times 10^{.75}$

38. $10^{1.250} \times 10^1$

39. $10^{2.73} \times 10^{1.24}$

40. $10^{.301} \times 10^{.477}$

41. $10^{2.53} \div 10^1$

42. $10^{.75} \div 10^{.25}$

43. $10^{1.36} \div 10^{.45}$

44. $10^{.7} \div 10^{.8}$

45. $10^{9.27} \times 10^{.84}$

46. $(10^2)^{\frac{1}{2}}$

47. $(5^{2.4})^{\frac{1}{2}}$

48. $(10^{2.25})^{\frac{1}{3}}$

49. $(10^{1.452})^4$

50. $(10^{9.762})^{\frac{1}{3}}$

51. $2^x \cdot 2^y$

52. $2^{2a} \cdot 2^a$

53. $4^c \div 4^d$

54. $3^n \div 3^m$

55. $5^x \div 5^{2x}$

56. $3^x \cdot 3^{3x} \div 3^{2x}$

57. $10^{1.25} \times 100$

58. $10^{2.1} \div 100$

59. $10^{1.50} \div 100$

60. $10^{2.75} \times 1000$

OPERATIONS WITH RADICALS (*Optional*)

188. When an indicated root can be obtained, the result is a **rational** number; when the root cannot be obtained, the result is an **irrational number,** or **radical.**

The denominator of the fractional exponent, or the small number written in the angle of the radical sign, is the **index** of the radical.

Thus the index of $\sqrt{5}$ is 2; of $\sqrt[3]{4}$ is 3; of $2^{\frac{1}{3}}$ is 3.

189. To manipulate radicals easily, remember:

I. The *definition:* $(\sqrt[n]{x})^n = x.$ Thus $(\sqrt[3]{5})^3 = 5.$

II. The *principle:* $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$; or $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}}$

III. The *principle:* $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$; or $\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$.

IV. *That radicals can be indicated by fractional exponents, which obey all the laws of exponents.*

190. Simplifying a radical:

(*a*) *A radical is usually written with as low an index as possible.*

Thus: $\sqrt[4]{64} = (8^2)^{\frac{1}{4}} = 8^{\frac{1}{2}} = \sqrt{8}.$

(*b*) *Factors are removed from the radicand when possible.*

Thus: $\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$

$$\sqrt[3]{72\,x^2y^4} = \sqrt[3]{8 \cdot y^3 \cdot 9\,x^2} = \sqrt[3]{8\,y^3} \cdot \sqrt[3]{9\,x^2} = 2\,y\sqrt[3]{9\,x^2}.$$

(*c*) *The radicand is written without a denominator.*

Thus: $$\sqrt{\frac{5}{12\,a}} = \sqrt{\frac{5 \cdot 3\,a}{12\,a \cdot 3\,a}} = \frac{\sqrt{15\,a}}{\sqrt{36\,a^2}} = \frac{\sqrt{15\,a}}{6\,a}$$

$$\sqrt[3]{\frac{3}{4\,a^2}} = \sqrt[3]{\frac{3 \cdot 2\,a}{8\,a^3}} = \frac{\sqrt[3]{6\,a}}{\sqrt[3]{8\,a^3}} = \frac{\sqrt[3]{6\,a}}{2\,a}$$

Observe: If the radical is a square root, the denominator is made a perfect square; if it is a cube root, the denominator is made a perfect cube; etc.

191. Rule. **To reduce a radical to its simplest form:**

(*a*) *Change it to an equal radical with lowest possible index.*

(*b*) *Remove from under the radical any perfect power factors.*

(*c*) *Change the radical to one in which the radicand is not fractional, as in § 190, c.*

EXERCISE 106, *a*

Simplify the following radicals:

1. $\sqrt[4]{81}$	3. $\sqrt[4]{36}$	5. $\sqrt[6]{16}$	7. $\sqrt[4]{25c^4}$
2. $\sqrt[4]{49}$	4. $\sqrt[6]{64}$	6. $\sqrt[6]{25}$	8. $\sqrt[4]{9a^2b^2}$
9. $\sqrt{18}$	11. $\sqrt[3]{24}$	13. $\sqrt[3]{72}$	15. $\sqrt{28a^2b}$
10. $\sqrt{200}$	12. $\sqrt[3]{88}$	14. $\sqrt[4]{32}$	16. $\sqrt[3]{24x^3}$
17. $\sqrt{\frac{2}{5}}$	19. $\sqrt{\frac{3a}{8b}}$	21. $\sqrt[3]{\frac{7}{2}}$	23. $\sqrt{\frac{5x}{18y^2}}$
18. $\sqrt{\frac{7}{12}}$	20. $\sqrt[3]{\frac{2}{3}}$	22. $\sqrt[3]{\frac{2}{5}}$	24. $\sqrt[3]{\frac{2}{3}a^2}$
25. $\sqrt{12x^3}$	27. $\sqrt[3]{16t^6}$	29. $\sqrt[3]{\frac{1}{27}m^2}$	31. $\sqrt[3]{\frac{4}{27}x^5y^2}$
26. $\sqrt[3]{27a^2b^3}$	28. $\sqrt[3]{27m^4}$	30. $\sqrt{98rs^3}$	32. $\sqrt{32m^4}$
33. $\sqrt{\frac{11}{32}}$	35. $\sqrt{\frac{3}{5x}}$	37. $\sqrt[3]{\frac{2a^3}{4}}$	39. $\sqrt[3]{\frac{4c^2}{9t}}$
34. $\sqrt[3]{\frac{5}{16}}$	36. $\sqrt[3]{\frac{3a}{4b}}$	38. $\sqrt{\frac{xy}{200}}$	40. $\sqrt{\frac{c-d}{c+d}}$

EXERCISE 106, *b*

1. $\sqrt[3]{54y^4}$	6. $\sqrt[4]{225}$	11. $\sqrt[3]{128m^4}$	16. $\sqrt[3]{\frac{8}{125}xy}$
2. $\sqrt[4]{8c^3d^5}$	7. $\sqrt[6]{64}$	12. $\sqrt[3]{108a^7}$	17. $\sqrt[3]{16x^3m^2}$
3. $\sqrt[4]{\frac{1}{81}x^2y^7}$	8. $\sqrt[4]{27}$	13. $\sqrt[4]{\frac{2}{81}ax^8}$	18. $\sqrt[4]{80c^3d^5}$
4. $\sqrt[5]{32x^5y^6}$	9. $\sqrt[6]{144}$	14. $\sqrt[4]{\frac{5}{256}x^4}$	19. $\sqrt[6]{128a^7}$
5. $\sqrt[6]{64x^7y}$	10. $\sqrt[8]{64}$	15. $\sqrt[5]{243x^2}$	20. $\sqrt[6]{xy^7z}$
21. $\sqrt[4]{\frac{3}{8x^3}}$	22. $\sqrt[4]{\frac{bx}{a}}$	23. $\sqrt[5]{\frac{a}{16b^4}}$	24. $\sqrt[6]{\frac{a^2y^7}{32x^6}}$

192. Addition and subtraction of radicals.

The sum of two radicals like $\sqrt{2}$ and $\sqrt{3}$ can be indicated thus: $\sqrt{2} + \sqrt{3}$. Or, each can be expressed decimally and the approximate sum be obtained.

Thus: $\sqrt{2} + \sqrt{3} = 1.414 + 1.732$, or 3.146.

In order that radicals may be combined, they must be similar radicals. Similar radicals are radicals which have the same index and which have the same radicand.

Example 1. $\sqrt{24} + 3\sqrt{6} = \sqrt{4 \cdot 6} + 3\sqrt{6} = 2\sqrt{6} + 3\sqrt{6}$, or $5\sqrt{6}$

Example 2. $\sqrt[3]{\frac{1}{4}} - \sqrt[3]{54} = \sqrt[3]{\frac{2}{8}} - \sqrt[3]{27 \cdot 2} = \frac{1}{2}\sqrt[3]{2} - 3\sqrt[3]{2}$, or $-2\frac{1}{2}\sqrt[3]{2}$

EXERCISE 107

Simplify and combine similar radicals:

1. $\sqrt{32} + \sqrt{72}$
2. $2\sqrt{18\,x} - \sqrt{50\,x}$
3. $\sqrt{128\,a} - 3\sqrt{8\,a}$
4. $\sqrt{\frac{1}{8}} + \frac{1}{4}\sqrt{18}$
5. $\sqrt{75\,m} - \sqrt{27\,m}$
6. $5\sqrt{\frac{1}{3}\,x^3} - \sqrt{\frac{4}{3}\,x^3}$
7. $3\,a\sqrt{16\,a} - \sqrt{144\,a^3}$
8. $2\,x\sqrt{25\,xy^2} - 3\,y\sqrt{4\,x^3y}$
9. $n\sqrt{8\,m^3n} - m\sqrt{32\,mn^3}$
10. $\sqrt[3]{16} + \sqrt[3]{2}$
11. $\sqrt[3]{54\,x} - \sqrt[3]{16\,x}$
12. $\sqrt[3]{6\,y^2} - \sqrt[3]{24\,y^2}$
13. $\sqrt[3]{\frac{2}{3}} + \sqrt[3]{\frac{9}{4}}$
14. $\sqrt[3]{\frac{1}{25}\,a} - \sqrt[3]{\frac{5}{27}\,a}$
15. $3\,x\sqrt[3]{\frac{1}{9}\,x} + 5\sqrt[3]{3\,x^4}$
16. $\sqrt[4]{81} + \sqrt[4]{16}$
17. $\sqrt[4]{48} - \sqrt[4]{3}$
18. $\sqrt[5]{2} + \sqrt[5]{64}$
19. $x^2\sqrt[6]{\dfrac{2}{x^5}} - \sqrt[6]{2\,x^7}$
20. $\sqrt{x^5} + \sqrt{x}$
21. $\sqrt{8\,x^2} - x\sqrt{18} + \sqrt{50\,x^2}$
22. $y\sqrt{108} - \sqrt{243\,y^2} + y\sqrt{75}$
23. $3\sqrt{11} - \sqrt{44} + \sqrt{99}$
24. $5\sqrt{\frac{1}{2}} + \frac{1}{4}\sqrt{8} - \sqrt{\frac{25}{2}}$
25. $\sqrt{48} - 3\sqrt{12} + 5\sqrt{\frac{1}{3}}$
26. $\sqrt{2\,a^2} - \sqrt{18\,a^2} - \frac{1}{2}\,a\sqrt{8}$
27. $3\sqrt{\frac{9}{2}\,a} - \sqrt{8\,a} + 2\sqrt{18\,a}$
28. $\sqrt[3]{16\,x} - \sqrt[3]{54\,x} + 2\sqrt[3]{2\,x}$
29. $2\sqrt[3]{\frac{2}{9}\,x} - \sqrt[3]{\frac{3}{4}\,x} - \sqrt[3]{6\,x}$
30. $\sqrt[4]{rs^4} - \sqrt[4]{r^4s} + r\sqrt[4]{16\,s}$

193. Multiplication of radicals.

Example 1. $2\sqrt{3} \cdot 3\sqrt{6} = 6\sqrt{3 \cdot 6} = 6\sqrt{3^2 \cdot 2} = 6 \cdot 3\sqrt{2} = 18\sqrt{2}.$

Example 2. $4\sqrt[3]{2} \cdot 5\sqrt[3]{4} = 20\sqrt[3]{2 \cdot 4} = 20\sqrt[3]{8} = 20 \cdot 2 = 40.$

Example 3. $\sqrt{3} \cdot \sqrt[3]{3} = 3^{\frac{1}{2}} \cdot 3^{\frac{1}{3}} = 3^{\frac{3}{6}} \cdot 3^{\frac{2}{6}} = (3^3 \cdot 3^2)^{\frac{1}{6}} = \sqrt[6]{3^5}.$

EXERCISE 108, *a*

Find the following products:

1. $\sqrt{3} \cdot \sqrt{18}$
2. $\sqrt{2} \cdot \sqrt{14}$
3. $2\sqrt{7} \cdot \sqrt{21}$
4. $3\sqrt{6} \cdot 2\sqrt{2}$
5. $3\sqrt{3} \cdot \sqrt{27}$
6. $\sqrt{5} \cdot \sqrt{15}$
7. $2\sqrt{7} \cdot 3\sqrt{7}$
8. $\sqrt{12} \cdot 3\sqrt{3}$
9. $\sqrt{2x} \cdot \sqrt{3x}$
10. $(\sqrt{5x})^2$
11. $(2\sqrt{3y})^2$
12. $(3\sqrt{t})^2$
13. *Illustrative Example*: $(2 + \sqrt{3})(3 - \sqrt{3})$
$= 6 + 3\sqrt{3} - 2\sqrt{3} - 3$, or $3 + \sqrt{3}$.
14. $(2 - \sqrt{5x})(2 + \sqrt{5x})$
15. $(7 - \sqrt{3y})(7 + \sqrt{3y})$
16. $(\sqrt{5a} + 3)(\sqrt{5a} - 3)$
17. $(2\sqrt{3} - 1)(2\sqrt{3} + 1)$
18. $(3 + \sqrt{3})(3 - \sqrt{3})$
19. $(2\sqrt{3} + 1)(\sqrt{3} - 1)$
20. $(3\sqrt{2} + 2)(\sqrt{2} - 3)$
21. $(5 - 2\sqrt{3})(1 + 2\sqrt{3})$
22. $(4 - \sqrt{5})(2 + \sqrt{5})$
23. $(\sqrt{3} + 1)(\sqrt{3} + 1)$
24. Does $1 + \sqrt{2}$ satisfy the equation $x^2 - 2x - 1 = 0$?
25. Does $\sqrt{3} - 1$ satisfy the equation $x^2 + 2x = 2$?
26. Is $2 - \sqrt{10}$ a root of $x^2 - 4x = 6$?

EXERCISE 108, *b*

Find:

1. $(\sqrt{x+1} - 2)^2$
2. $(\sqrt{b-3} + 4)^2$
3. $(5 - \sqrt{x-1})^2$
4. $(2 + \sqrt{x-5})^2$
5. $(\sqrt{y-3} - 2)^2$
6. $(\sqrt{x} - \sqrt{x+1})^2$
7. $\sqrt[3]{9} \cdot \sqrt[3]{6}$
8. $\sqrt[3]{16} \cdot \sqrt[3]{12}$
9. $\sqrt[3]{3x} \cdot \sqrt[3]{9x^2}$
10. $\sqrt[3]{6x^3} \cdot \sqrt[3]{9}$
11. $\sqrt[4]{3} \cdot \sqrt[4]{27}$
12. $\sqrt[4]{6x} \cdot \sqrt[4]{8x^3}$
13. $\sqrt[5]{4x} \cdot \sqrt[5]{8x^4}$
14. $\sqrt[6]{4} \cdot \sqrt[6]{16}$
15. $\sqrt{2} \cdot \sqrt[3]{4}$
16. $\sqrt{x} \cdot \sqrt[4]{x^3}$
17. $\sqrt{10} \cdot \sqrt[3]{25}$
18. $\sqrt[3]{9} \cdot \sqrt{15}$

194. Division of radicals.

Example 1. $\sqrt{27} \div \sqrt{9} = \sqrt{27 \div 9} = \sqrt{3}.$

Example 2. $3 \div \sqrt{2} = \dfrac{3}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{3\sqrt{2}}{2}.$

By multiplying the denominator by itself, we *rationalize* the denominator. We can now substitute the value of $\sqrt{2}$ and simplify.

Example 3. $\dfrac{\sqrt{2}}{\sqrt{2}-1} = \dfrac{\sqrt{2}(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = \dfrac{2+\sqrt{2}}{2-1} = 2+\sqrt{2}.$

In this example, the *difference* of two numbers in the denominator was multiplied by the sum; the product is *rational.* $(\sqrt{2}-1)$ and $(\sqrt{2}+1)$ are called **conjugate expressions.**

EXERCISE 109, *a*

Find the value of:

1. $\dfrac{\sqrt{21}}{\sqrt{7}}$
2. $\dfrac{\sqrt{20}}{\sqrt{5}}$
3. $\dfrac{\sqrt{50\,x}}{\sqrt{2\,x}}$
4. $\dfrac{a\sqrt{75}}{a\sqrt{3}}$
5. $\dfrac{\sqrt{3}}{\sqrt{\frac{1}{3}}}$
6. $\dfrac{\sqrt{x^3y^3}}{\sqrt{xy}}$
7. $\dfrac{\sqrt{30\,a^3}}{\sqrt{6\,a}}$
8. $\dfrac{\sqrt{24\,x^3}}{\sqrt{8\,x}}$
9. $\dfrac{\sqrt{14\,y^3}}{\sqrt{\frac{1}{7}y}}$
10. $\dfrac{2\sqrt{10}}{\sqrt{5}}$
11. $\dfrac{2}{\sqrt{3}}$
12. $\dfrac{3}{\sqrt{6}}$
13. $\dfrac{4}{\sqrt{12}}$
14. $\dfrac{9}{\sqrt{18}}$
15. $\dfrac{2\,x}{\sqrt{6\,x}}$
16. $\dfrac{1+\sqrt{2}}{1-\sqrt{2}}$
17. $\dfrac{\sqrt{3}-2}{\sqrt{3}+2}$
18. $\dfrac{2+\sqrt{5}}{3-\sqrt{5}}$
19. $\dfrac{\sqrt{6}-2}{\sqrt{6}+3}$
20. Find the value of $\dfrac{x-1}{x+1}$:

    (*a*) when $x = \sqrt{3}-1$; (*b*) when $x = 2-\sqrt{2}$.

EXERCISE 109, *b*

1. $\dfrac{\sqrt[3]{6}}{\sqrt[3]{2}}$
2. $\dfrac{\sqrt[3]{54}}{\sqrt[3]{2}}$
3. $\dfrac{\sqrt[3]{24}}{\sqrt[3]{3}}$
4. $\dfrac{\sqrt[3]{40}}{\sqrt[3]{5}}$
5. $\dfrac{\sqrt[3]{12}}{\sqrt[3]{3}}$
6. $\dfrac{\sqrt[3]{15}}{\sqrt[3]{5}}$
7. $\dfrac{\sqrt[4]{32}}{\sqrt[4]{2}}$
8. $\dfrac{\sqrt[5]{10}}{\sqrt[5]{2}}$
9. $\dfrac{6}{\sqrt[3]{4}}$
10. $\dfrac{10}{\sqrt[3]{25}}$

EXERCISE 110. CHAPTER MASTERY-TEST

1. Give the numerical value of:

(*a*) $100^{\frac{1}{2}}$ (*b*) $(-27)^{\frac{1}{3}}$ (*c*) $(\frac{1}{16})^0$ (*d*) 2^{-3} (*e*) $16^{-\frac{1}{4}}$

2. Give the simplest form of:

(*a*) $\sqrt{8x^3}$ (*b*) $\sqrt[3]{27xy^3}$ (*c*) $\sqrt[4]{2x^4y}$ (*d*) $\sqrt[3]{-\frac{1}{8}x^3y}$

3. Give the simplest form of:

(*a*) $\sqrt{\dfrac{3}{2a}}$ (*b*) $\sqrt{\dfrac{4}{3b^2}}$ (*c*) $\sqrt[3]{\dfrac{1}{9x^3}}$ (*d*) $\sqrt[3]{-\dfrac{8}{x^2y^3}}$

4. Simplify: (*a*) $\sqrt{72}-\sqrt{32}$ (*b*) $\sqrt[3]{24x}-\sqrt[3]{54x}$

5. Simplify: (*a*) $\sqrt{\dfrac{5}{a}}+\sqrt{3\frac{1}{5}a}$ (*b*) $\sqrt[3]{\frac{3}{4}x}-\sqrt[3]{\frac{2}{9}x}$

6. Find: (*a*) $\sqrt{3}\cdot\sqrt{6}$ (*b*) $2\sqrt{7x}\cdot 3\sqrt{14x}$

7. Find: (*a*) $(\sqrt{2}+x)(\sqrt{2}-x)$ (*b*) $(\sqrt{7}-3y)(\sqrt{7}+2y)$

8. Does $2-\sqrt{3}$ satisfy the equation $x^2-4x-3=0$?

9. Find: (*a*) $\dfrac{6\sqrt{20}}{3\sqrt{5}}$ (*b*) $\dfrac{9}{\sqrt{6}}$ (*c*) $\dfrac{\sqrt[3]{15}}{\sqrt[3]{3}}$ (*d*) $\dfrac{\sqrt[4]{48}}{\sqrt[4]{3}}$

10. Find the value of $\dfrac{2x-1}{x+2}$ when $x=\sqrt{2}-3$.

11. Perform the indicated operations.

(*a*) $(s^{-3})(s^{+2})$ (*c*) $(y^{\frac{2}{3}})(y^{-\frac{1}{3}})$ (*e*) $10^{2.5}\times 100$

(*b*) $(x^{\frac{1}{3}})(x^{\frac{1}{2}})$ (*d*) $(m^{-\frac{1}{4}})(m^0)$ (*f*) $(x^{-2}+y^{-2})(x^{-2}+y^{-2})$

12. Perform the indicated operations.

(*a*) $(t^{-4})\div(t^{-5})$ (*c*) $(y^{+\frac{1}{2}})\div(y^{\frac{1}{4}})$ (*e*) $3^{2x}\div 3^x$

(*b*) $(x^{+2})\div(x^{-3})$ (*d*) $(z^{-\frac{2}{3}})\div(z^{+\frac{1}{3}})$ (*f*) $10^{1.46}\div 10^{.73}$

13. Give the simplest form for:

(*a*) $(10^2)^3$ (*c*) $(10^{-1.3})^2$ (*e*) $(x^{\frac{2}{3}})^3$ (*g*) $(x^2y^4)^{-\frac{1}{2}}$

(*b*) $(10^{1.5})^2$ (*d*) $(x^3)^{-2}$ (*f*) $(y^6)^{\frac{1}{3}}$ (*h*) $(y^{-3}z^{-3})^{\frac{1}{3}}$

14. Give the simplest form for:

(*a*) $9^{\frac{3}{2}}$ (*c*) $16^{\frac{3}{4}}$ (*e*) $(-8)^{\frac{5}{3}}$ (*g*) $(\frac{1}{27}a^3)^{\frac{2}{3}}$

(*b*) $8^{\frac{4}{3}}$ (*d*) $25^{\frac{3}{2}}$ (*f*) $(-27)^{\frac{4}{3}}$ (*h*) $(-\frac{1}{8}x^6)^{\frac{4}{3}}$

EXERCISE 111. CUMULATIVE REVIEW

1. Find the prime factors of:

(*a*) $x^{12} - y^6$ (*c*) $y^3 - 6y^2 - 3y + 8$

(*b*) $x^{2+n} + 2x^{1+n}y + x^n y^2$ (*d*) $x^5 + 32y^5$

2. Divide $6a^{-3} - a^{-2} - 27a^{-1} + 20$ by $3a^{-1} - 5$. Express the quotient with positive powers of a.

3. Simplify $15^0 \cdot \sqrt{18x^{2n}} \cdot (3x^{n+1})^{-1}$.

4. Rationalize the denominator (or perform the indicated division):

(*a*) $\dfrac{4m}{\sqrt{2m}}$ (*b*) $\dfrac{3a}{\sqrt[3]{6a}}$ (*c*) $\dfrac{\sqrt{6}}{\sqrt{3} - \sqrt{2}}$.

5. Solve for x to the nearest hundredth: $x^2 - 2.5x = 1.5$.

6. Solve the following system; group your results.

$$\begin{cases} x - y = 1 \\ x^2 + xy + y^2 = 37 \end{cases}$$

7. Solve and check: $\sqrt{9x^2 + 5} - 3x = 1$.

8. (*a*) Without solving the equation determine the nature of the roots of $5x^2 - 4x - 3 = 0$.

(*b*) What is the sum of the roots? The product?

9. Determine graphically the roots of the equation $x^2 - 5x = 2$ correct to the nearest tenth.

10. Determine graphically the common solution of:

(*a*) The system $\begin{cases} 2x + y = 8 \\ 3x - y = 2 \end{cases}$ (*b*) The system $\begin{cases} x - y = 3 \\ xy = 10 \end{cases}$

11. A flower garden contains 1500 square feet. It is surrounded by a path which is 3 feet wide. The area of the path is 516 square feet. What are the dimensions of the flower garden?

12. $Ax^2 + By^2 = 1$ is the equation of a graph which goes through the points $x = 4\sqrt{2}$; $y = 3$; and also the point $x = 2$, $y = 6\sqrt{2}$. Find A and B.

XIII. LOGARITHMS

195. Logarithms are special exponents.

Every positive number can be expressed exactly or approximately as a power of 10; as $100 = 10^2$.

The exponent required is called the **logarithm of the number to the base 10.**

Thus 2 is the logarithm of 100 to the base 10.
It is written briefly thus: $2 = \log_{10} 100$, or $2 = \log 100$.

196. How some logarithms can be obtained.

$10^0 = 1$, or $0 = \log 1$. $\qquad 10^1 = 10$, or $1 = \log 10$.

$10^{.5} = \sqrt{10} = 3.162$; or $.50 = \log 3.162$.

$10^{.25} = (10^{.5})^{\frac{1}{2}} = \sqrt{10^{.5}} = \sqrt{3.162} = 1.778$; or $.25 = \log 1.778$.

By similar computation, the table below can be obtained.

197. How logarithms are used.

Example 1. Find 3.1623×17.782.

Solution. 1. $\quad 3.1623 \times 17.782$
2. $\quad = 10^{.50} \times 10^{1.25}$
3. $\quad = 10^{1.75}$
4. $\quad = 56.234$ (in the table)
5. $\quad \therefore 3.1623 \times 17.782 = 56.234$.

The solution is approximately correct.

Example 2. Find $(5.6234)^2 \times 31.623 \div 17.782$.

Solution. 1. $(5.6234)^2 \times 31.623 \div 17.782$
2. $\quad = (10^{.75})^2 \times 10^{1.50} \div 10^{1.25}$
3. $\quad = 10^{1.50+1.50-1.25} = 10^{1.75}$
4. $\quad \therefore (5.6234)^2 \times 31.623 \div 17.782 = 56.234$

$10^{.00}$	$= 1.0000$
$10^{.25}$	$= 1.7782$
$10^{.50}$	$= 3.1623$
$10^{.75}$	$= 5.6234$
$10^{1.00}$	$= 10.0000$
$10^{1.25}$	$= 17.7820$
$10^{1.50}$	$= 31.6230$
$10^{1.75}$	$= 56.2340$
$10^{2.00}$	$= 100.0000$

This solution also may be checked by ordinary computation by *one who has a lot of time and ambition.*

198. Logarithms of numbers to the base 10 are called **common logarithms** or simply logarithms.

199. Graph of the functional relation y = log x.

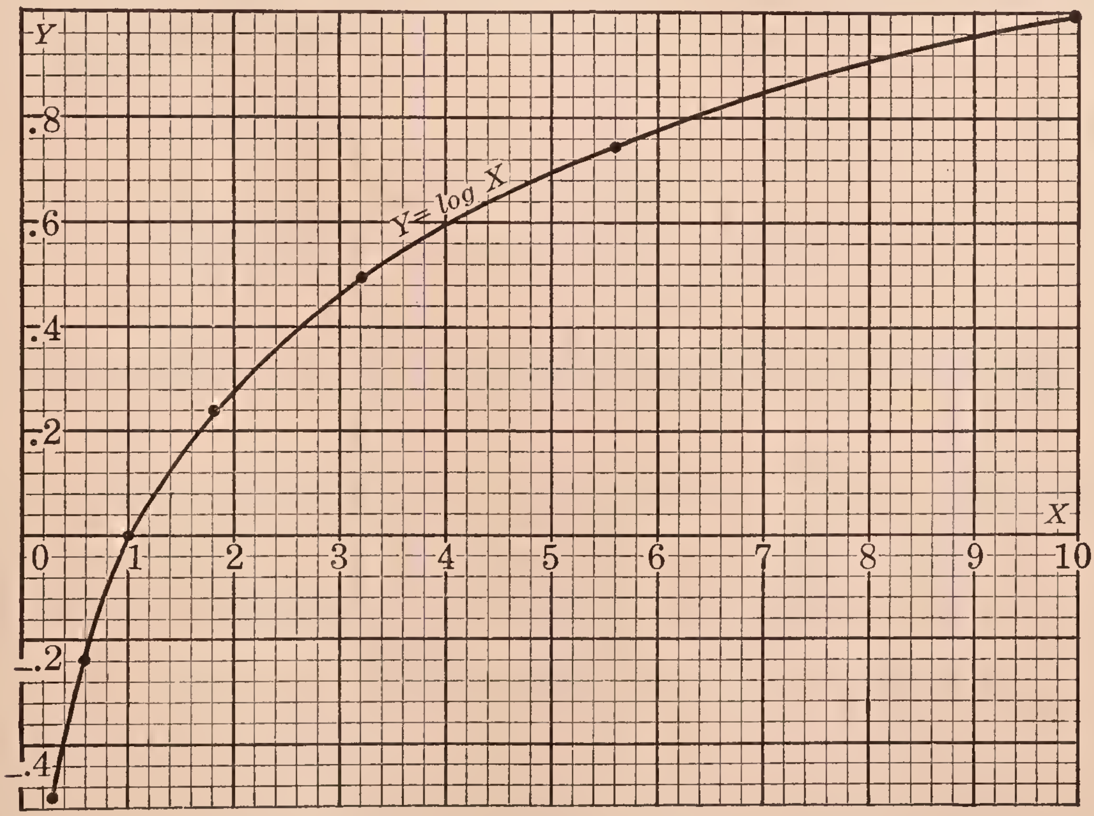
Besides the values of the table on page 184, observe:

$$10^{-.25} = 10^{.75-1.00} = 10^{.75} \div 10^{1} = 5.6234 \div 10 = .562.$$

Similarly $10^{-.50} = 10^{.5} \div 10^{1} = 3.1623 \div 10 = .3162.$

Express the values of x correct to tenths, and the values of y correct to hundredths.

When x =	.3	.6	1.0	1.8	3.2	5.6	10
then y =	− .50	− .25	0.00	.25	.50	.75	1.00



There are no values of y for negative values of x.

y is negative when x lies between 0 and 1.

y increases as x increases.

200. Most numbers are not exact powers of 10.

Thus, from the graph on page 185, log 6 = about .78

Correct to four decimal places log 6 = .7782. Similarly the log 60 is 1.7782.

The integral part of a logarithm is called the **characteristic** and the decimal part the **mantissa.**

Thus, the characteristic of log 60 is 1, and the mantissa is .7782.

201. Finding the characteristic of the logarithm of a number greater than 1.

It is known that $3.53 = 10^{.5478}$ $\quad\therefore \log 3.53 = .5478$

$35.3 = 10 \times 3.53 = 10 \times 10^{.5478} = 10^{1.5478} \quad \therefore \log 35.3 = 1.5478$

$353 = 10 \times 35.3 = 10 \times 10^{1.5478} = 10^{2.5478} \quad \therefore \log 353 = 2.5478$

In this last line, 353 has three figures to the left of the decimal point; its logarithm has characteristic 2, which is 1 less than 3.

Rule. *The characteristic of the common logarithm of a number greater than 1 is one less than the number of significant figures to the left of the decimal point.*

Thus, the characteristic of log 357.83 is 2; of log 70390 is 4.

202. Finding the characteristic of the logarithm of a number less than 1.

$$.353 = \frac{3.53}{10} = \frac{10^{.5478}}{10} = 10^{.5478-1} \qquad \therefore \log .353 = .5478 - 1$$

$$.0353 = \frac{.353}{10} = \frac{10^{.5478-1}}{10} = 10^{.5478-2} \qquad \therefore \log .0353 = .5478 - 2$$

$$.00353 = \frac{10^{.5478-2}}{10} = 10^{.5478-3} \qquad \therefore \log .00353 = .5478 - 3$$

Observing you will find that the following rule is correct.

Rule. *The characteristic of the common logarithm of a (positive) number less than 1 is negative; numerically it is one more than the number of zeros between the decimal point and the first significant figure.*

203. The method of writing a negative characteristic.

In § 202 log .353 = .5478 − 1. Actually, therefore, log .353 is − .4522, a negative number. However, the positive mantissa and the negative characteristics are retained, as follows.

.5478 − 1 is written: 9.5478 − 10. Numerically the two expressions have equal value. Note that 9 − 10 = − 1.

In general, *decide upon the characteristic by the rule in* § 202; then, if it is − 1, write it 9 − 10; if − 2, write it 8 − 10; etc.; *and then omit the* − 10, *usually.*

Thus, log .02 is .3010 − 2, or 8.3010 − 10, or 8.3010.

NOTE. The negative characteristic is often written thus: log .02 = $\bar{2}.3010$; again, log .353 = $\bar{1}.5478$. The minus sign is written over the characteristic to indicate that it alone is negative, the mantissa being positive. Your teacher will decide which of these two methods you will use.

EXERCISE 112

What is the characteristic of the logarithm of:

1. 59	**5.** 72,860	**9.** 5.08	**13.** 984.2
2. 540	**6.** 11.2	**10.** 3002	**14.** 87,600
3. 4000	**7.** 367.2	**11.** 21.67	**15.** 2.193
4. 8	**8.** 50900	**12.** 100.5	**16.** 1,000,000

Tell the number of significant figures preceding the decimal point when the characteristic of the logarithm is:

17. 5 **18.** 3 **19.** 0 **20.** 1 **21.** 4 **22.** 2

Write in two ways the characteristic of the logarithm of:

23. .5	**25.** .07	**27.** .6432	**29.** .1007
24. .004	**26.** .01003	**28.** .04216	**30.** .00008

Tell the number of zeros preceding the first significant figure when the characteristic of the logarithm is:

31. − 2 **32.** − 4 **33.** − 1 **34.** − 5 **35.** − 3

204. The mantissa of the logarithm of a number. From the illustrations in § 201 to § 203, it is clear that *the common logarithms of all numbers having the same significant figures have the same mantissas.* These mantissas are given in a table of logarithms such as appears on pages 254 and 255.

205. Finding the logarithm of a three digit number.

Example 1. Find the logarithm of 16.8.

Solution. 1. In the column headed "No." (page 254) find 16. On the horizontal line opposite 16, pass over to the column headed by the figure 8. The mantissa 2253 found there is the required mantissa.

2. The characteristic is 1, by the rule in § 201.

3. $\therefore$ log 16.8 is 1.2253.

Rule. To find the logarithm of a number of three figures:

1. *Look in the column headed "No." (pages 254–255) for the first two figures of the given number. The mantissa will be found on the horizontal line opposite these two figures and in the column headed by the third figure of the given number.*

2. *Prefix the characteristic according to* § 201 *and* § 202.

Example 2. Find log .304.

Solution. 1. Opposite 30 in the column headed by 4 is the mantissa .4829. The characteristic is -1 or $9 - 10$. (§ 202 and § 203.)

2. $\therefore \log .304 = 9.4829 - 10, = 9.4829.$

NOTE. The logarithm of a number of one or two digits may be found by using the column headed 0. Thus the mantissa of log 8.3 is the same as the mantissa of log 8.30; of log 9, the same as of log 900.

EXERCISE 113

Find the logarithm of:

1. 365	**6.** 64	**11.** .841	**16.** .000834
2. 571	**7.** 9	**12.** .0628	**17.** .07
3. 847	**8.** 5.2	**13.** .00175	**18.** 3.14
4. 902	**9.** 43.6	**14.** 7680	**19.** 40.8
5. 200	**10.** 720	**15.** 25900	**20.** .16

206. The logarithm of a number of more than three digits.

Example 1. Find log 327.5.

Solution. 1.	log 327 = 2.5145	Difference
From the table	log 327.5 = ?	= .0014.
on page 254.	log 328 = 2.5159	

2. Since 327.5 is between 327 and 328, its logarithm must be between their logarithms. An increase of one unit in the number (from 327 to 328) produces an increase of .0014 in the mantissa. It is *assumed* therefore that an increase of .5 in the number (from 327 to 327.5) produces an increase of .5 of .0014, or of .0007 in the mantissa.

3. $\therefore$ log 327.5 = 2.5145 + .0007, or 2.5152.

.0014 is called the *tabular difference*. The zeros are usually omitted.

Example 2. Find log 34.67.

Solution. 1. Mantissa of log 346 = 5391
Mantissa of log 347 = 5403

2. Tabular difference = 12. $.7 \times 12 = 8.4$ or 8

3. $\therefore$ mantissa for log 3467 = 5391 + 8, or 5399

4. $\therefore$ log 34.67 = 1.5399.

Rule. 1. *Find the mantissa for the first three figures, and the tabular difference for that mantissa.*

2. *Multiply the tabular difference by the remaining figures of the given number, preceded by a decimal point.*

3. *Add the result of Step* 2 *to the mantissa obtained in Step* 1, *writing the sum correct to four places.*

4. *Prefix the proper characteristics.* (*See Rules, page* 186.)

EXERCISE 114

Find the logarithm of:

1. 342.5	**6.** 501.6	**11.** .04873	**16.** 1.067
2. 252.1	**7.** 28.25	**12.** 328.2	**17.** 5.243
3. 865.2	**8.** 1.158	**13.** 453.3	**18.** 3.142
4. 764.4	**9.** 7.631	**14.** 86.43	**19.** 632.4
5. 438.3	**10.** .5842	**15.** 3.728	**20.** 82.56

207. Finding the number when its logarithm is given.

NOTE. Some teachers call this finding the *antilogarithm.*

Example 1. Find the number whose logarithm is 1.6571.

Solution. 1. Find the mantissa 6571 in the table on pages 254–255.

2. In the column headed "No." on the line with 6571 is 45. At the head of the column containing 6571 is 4. Hence the number sought has the figures 454.

3. The characteristic being 1, the number must have two figures to the left of the decimal point. ∴ the number is 45.4.

Example 2. Find the number whose logarithm is 1.3934.

Solution. 1. The mantissa 3934 *does not appear* in the table.

The next less mantissa is 3927, and the next greater is 3945.

$$\text{That is:}\quad \begin{array}{l} \text{mantissa of log } 247 = 3927 \\ \text{mantissa of log } x = 3934 \\ \text{mantissa of log } 248 = 3945 \end{array} \left.\begin{array}{l}\left.\begin{array}{l}\\ \\ \end{array}\right\} \begin{array}{c}\text{Difference} \\ = 7.\end{array} \\ \\ \end{array}\right\} \begin{array}{l}\text{Tabular} \\ \text{difference} \\ = 18.\end{array}$$

2. The increase of 18 in the mantissa produces an increase of 1 in the number. We *assume* that the increase of 7 produces an increase of $\frac{7}{18}$ or about .4 in the number.

3. Hence the number has the figures 247.4.

4. Since the characteristic is 1, the number is 24.74.

Example 3. Find the antilogarithm of 9.3940.

Solution. 1. Again x is between 247 and 248.

2. The difference = 13. Tabular difference = 18. $\frac{13}{18} = .7$

3. ∴ the number has the figures 247.7.

4. Since the characteristic is 9, the number is .2477.

EXERCISE 115

Find the number whose logarithm is

1. 2.7408	**6.** 9.5969(– 10)	**11.** 2.2930	**16.** 8.9194
2. 1.6678	**7.** 8.3429(– 10)	**12.** 1.9665	**17.** 7.7978
3. .4188	**8.** 3.8497	**13.** 3.8598	**18.** 3.2306
4. 3.8983	**9.** 7.3288(– 10)	**14.** 9.7606	**19.** 1.8817
5. 2.9417	**10.** .1195	**15.** 4.3346	**20.** 2.9986

The Operations with Logarithms

208. The logarithm of a product *equals the sum of the logarithms of the factors.*

That is: $\log MN = \log M + \log N$

Proof 1. Let $M = 10^x$; or $x = \log M$
and let $N = 10^y$; or $y = \log N$

2. $MN = 10^{x+y}$

3. $\therefore \log MN = x + y$; or $\log MN = \log M + \log N$.

Example. Find the value of $7.208 \times .0631$.

Solution. 1. Let $v = 7.208 \times .0631$.
2. $\therefore \log v = \log 7.208 + \log .0631$
3. $\therefore \log v = 9.6578$
4. $\therefore v = .6573$

log 7.208 =	.8578
log .0631 =	8.8000 − 10
	9.6578 − 10

By the suggestion on page 68, it may be preferable to write this as 657, since there are only three significant figures in .0631.

At any rate, do not keep more than four significant figures.

EXERCISE 116

Find the following products by logarithms:

1. 42.5×26.8
2. 3.89×72.6
3. $535 \times .621$
4. $.342 \times 2.15$
5. $.654 \times .368$
6. 239.5×38.4
7. 871.2×45
8. 1.414×360
9. $8.42 \times .793$
10. $6.282 \times .778$
11. 7.026×8059
12. 432.4×1.658
13. 93.62×768.7
14. $84.75 \times .036$
15. $76.85 \times .0043$
16. $C = 2\pi r$. Find C when $\pi = 3.142$ and $r = 13$.
17. $S = \pi rh$. Find S, when $\pi = 3.142$; $r = 11$; $h = 16$.
18. $I = prt$. Find I, when $p = \$2750$; $r = .06$; $t = 4.5$.
19. $\text{Log } x = .4771$. Find $\log x^2$.
20. $\text{Log } y = .3010$. Find $\log 10y$.
21. Find $84.75 \times .368 \times 3.14$ by logarithms.
22. Find by logarithms the value of $(73.84)^2$.

209. The logarithm of the quotient *of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

That is: $\log (M \div N) = \log M - \log N$.

Proof 1. Let $M = 10^x$, or $x = \log M$
and let $N = 10^y$, or $y = \log N$

2. $\therefore M \div N = 10^{x-y}$

3. $\therefore \log(M \div N) = x - y$; or $\log(M \div N) = \log M - \log N$.

Example 1. Find the value of $215 \div 7.25$.

Solution. 1. Let $v = 215 \div 7.25$
2. $\therefore \log v = \log 215 - \log 7.25$
3. $\therefore \log v = 1.4721$
4. $\therefore v = 29.65$
or $v = 29.6$. (See p. 68.)

$$\begin{aligned} \log 215 &= 2.3324 \\ \log 7.25 &= .8603 \\ \text{Subtract}\ & \overline{1.4721} \end{aligned}$$

Find v by the rule on page 190.

Example 2. Find the value of $.192 \div .216$.

Solution. 1. $\log v = \log .192 - \log .216$
2. $\log v = 9.8488$
3. $v = .706$.

$$\begin{aligned} \log .192 &= 19.2833 - 20 \\ \log .216 &= 9.3345 - 10 \\ & \overline{9.8488 - 10} \end{aligned}$$

Note. $\log .192 = 9.2833 - 10$. If we subtract 9.3345 from this, we get a negative result. Since the mantissas in the table are all positive numbers, we write $9.2833 - 10$ as $19.2833 - 20$.

210. Optional method for division. Observe that

$$M \div N = M \times \left(\frac{1}{N}\right); \text{ so } \log M \div N = \log M + \log\left(\frac{1}{N}\right)$$

The $\log\left(\frac{1}{N}\right)$ is called the **cologarithm** of N.

Now $\log\left(\frac{1}{N}\right) = \log 1 - \log N = 0 - \log N = 10 - \log N - 10$.

The advantage is that *cologarithms are added.*

$$\begin{aligned} \text{Thus: } \log \frac{75}{26 \times 1.8} = \log 75 \quad &= 1.8751 \\ + \operatorname{colog} 26 &= 8.5850 - 10 \\ + \text{c} \log 1.8 &= 9.7447 - 10 \\ \therefore \log \frac{75}{26 \times 1.8} \quad &= 20.2048 - 20. \end{aligned}$$

Colog 26 =
$10 - 1.4150 - 10$
$= 8.5850 - 10$
Colog 1.8 =
$10 - .2553 - 10$
$= 9.7447 - 10.$

211. The logarithm of a power (or a root) of a number is the logarithm of the number multiplied by its exponent; that is

$$\log M^p = p \log M.$$

Proof 1. Let $M = 10^x$

2. $\therefore M^p = (10^x)^p = 10^{xp}$

3. $\therefore \log M^p = xp$

4. $\therefore \log M^p = p \log M.$

Example 1. Find by logarithms 1.04^{10}.

Solution. 1. $\log 1.04^{10} = 10 \log 1.04 = 10 \times .0170 = .1700$

2. $\therefore 1.04^{10} = 1.479.$

Example 2. Find by logarithms $\sqrt[4]{.0359}$.

Solution. 1. $\log \sqrt[4]{.0359} = \frac{1}{4} \log .0359$, or $\frac{1}{4}(8.5551 - 10)$

2. $\therefore \log \sqrt[4]{.0359} = \frac{1}{4}(38.5551 - 40)$ (See note below.)

3. $\therefore \log \sqrt[4]{.0359} = 9.6387 - 10$

4. $\therefore \sqrt[4]{.0359} = .4352.$

NOTE. To divide a negative logarithm, write it in such form that the negative part of the characteristic may be divided exactly by the divisor, and give -10 as quotient.

EXERCISE 117

Find by logarithms the value of:

1. $335 \div 56$

2. $483 \div 71$

3. $230.4 \div 125$

4. $739.8 \div 1.73$

5. $3305 \div 1.414$

6. $8.964 \div 45.25$

7. $\dfrac{4.16 \times 32}{485}$

8. $\dfrac{35.2 \times 1.52}{53.87}$

9. $\dfrac{43.57 \times .069}{3.14}$

10. $\dfrac{14.07 \times 347}{18}$

11. $\dfrac{3.25 \times .0063}{.007}$

12. $\dfrac{527.8 \times .069}{2.449}$

13. 323^2

14. 4.025^2

15. $\sqrt{418.5}$

16. $\sqrt[3]{784}$

17. $\sqrt[4]{92.04}$

18. 8.975^2

19. $\frac{4}{3} \times 3.142 \times 6.5^3$

20. 3.142×19^2

EXERCISE 118

Use logarithms to find products, quotients, powers, etc.

1. By the formula $I = prt$.
(*a*) Find I, if $p = \$4250$; $r = 4\%$; $t = 4$ yr.
(*b*) Find p if $I = \$375$; $r = 5\%$; $t = 3.5$ yr.
(*c*) Find t if $I = \$840$; $r = 6\%$; and $p = \$2200$.
(*d*) Find r if $I = \$750$; $p = \$3750$; and $t = 4$.

2. $z = 2\pi rh$. $(\pi = 3.142)$
(*a*) Find z when $r = 13$; and $h = 7.5$.
(*b*) Find r when $h = 11.2$; and $z = 628$.
(*c*) Find h when $z = 964$ and $r = 6.5$.

3. $t = \pi\sqrt{\frac{l}{g}}$. $(\pi = 3.142;\ g = 32.16)$
(*a*) Find t when $l = 30$.
(*b*) Find l when $t = 2$.

4. $V = \pi r^2 h$.
(*a*) Find V when $r = 12.4$ and $h = 30.3$.
(*b*) Find h when $V = 20250$ and $r = 15$.
(*c*) Find r if $h = 575$, and $V = 8550$.

5. $V = \frac{1}{3}\pi r^2 h$.
(*a*) Find V when $r = 6$ and $h = 13$.
(*b*) Find h when $r = 7.5$ and $V = 630$.
(*c*) Find r when $V = 725$ and $h = 12$.

6. $S = \frac{1}{2}gt^2$. $(g = 32.16)$
(*a*) Find S when $t = 4.5$.
(*b*) Find t when $S = 375$.

7. $A = p\left(1 + \frac{r}{100}\right)^n$ is the formula for the amount to which p dollars will accumulate at $r\%$, compounded annually for n years. (*a*) Find A if $p = \$450$; $r = 4\%$; $n = 8$.
(*b*) Find p if $A = 20000$; $r = 4\%$; $n = 10$.

EXERCISE 119. CUMULATIVE REVIEW

1. Find the prime factors of each of the following:

(*a*) $10a - 11a^2 - 6a^3$ (*d*) $x^5 - \frac{1}{32}y^5$

(*b*)* $x^3 + 2x^2 - 5x - 6$ (*e*) $225x^2 + 30mn - 9n^2 - 25m^2$

(*c*) $9x^{4a} - 12x^{2a} + 4$ (*f*) $9(a+b)^2 - 25$

2. Rationalize the denominator of $\dfrac{\sqrt{15}}{\sqrt{5}-\sqrt{3}}$

3. Solve and check the equation

$$3\sqrt{x+1} + x = 9$$

4. Find the roots of $.2x^2 - .32x = .4$ to the nearest hundredth.

5. Solve the system $\begin{cases} a^2 + ab + b^2 = 19 \\ a^2 - ab + b^2 = 7 \end{cases}$

Group your results, and check.

6. (*a*) Multiply $x^{\frac{2}{5}} - 2x^{\frac{1}{5}} + 3$ by $x^{\frac{1}{5}} - 1$.

(*b*) Express the result without any fractional exponents.

7. (*a*) Without solving the equation, determine the nature of the roots of the equation $4x^2 - 7x + 9 = 0$.

(*b*) Form the equation whose roots are $-\frac{1}{2}$ and $\frac{2}{3}$.

(*c*) For what value of k will the roots of $2x^2 - 5x + k = 0$ be equal?

8. By the use of logarithms, find the value of

$$\frac{(2.53) \times \sqrt{15.2}}{.8514}$$

9. A boat can go 15 miles upstream in the time that it requires to go 25 miles downstream. How does the rate of the boat in still water compare with the rate of the stream?

10. Solve the system $\begin{cases} x^2 = y \\ x + 2y = 6 \end{cases}$ graphically.

11. Determine graphically the roots of $3x^2 - 2x = 8$.

12. Determine graphically the roots of $3x^2 - 2x = 15$.

XIV. PROGRESSIONS

Arithmetic Progression

212. An **arithmetic progression** (A. P.) is a sequence of numbers, called **terms**, each of which after the first is derived from the preceding by adding to it a fixed number, called the **common difference**.

Thus, the amounts \$10.05; \$10.10; \$10.15; ⋯ form an A. P. These are the monthly payments for a washing machine, with interest at 6%. Each payment, clearly, is 5¢ more than the preceding.

Again, 9, 6, 3, ⋯ is an A. P. The common difference is -3. The next two terms are 0 and -3.

Note. The common difference is found by subtracting any term from the one following it.

EXERCISE 120

Determine which of the following are arithmetic progressions; determine the common difference and the next two terms of the arithmetic progressions:

1. 5, 8, 11, 14, ⋯
2. 3, 8, 13, 18, ⋯
3. 1, 5, 8, 13, ⋯
4. 32, 26, 20, 14, ⋯
5. $1\frac{2}{3}, 2\frac{1}{3}, 3, 3\frac{2}{3}, \cdots$
6. $6r, 8.5r, 11r, \cdots$
7. $3a, .5a, -2a, \cdots$
8. 1.05, 1.10, 1.15 ⋯
9. $c + d, 2c + d, 3c + d \cdots$
10. $6m + 8n, 4m + 9n, 2m + 10n, \cdots$

Write the first five terms of the A. P.	11	12	13	14	15
in which the first term is . . .	13	22	3.5	y	a
and the common difference is . .	6	-7	4.5	-5	d

16. Find the 15th term of the A. P. whose first term is 5, and common difference 7.

213. The *n*th term of an arithmetic progression. It is possible to determine a particular term of an arithmetic progression without finding all of the preceding terms.

Given the first term a, the common difference d, and the number of the term, n, of an arithmetic progression.

Find the nth term, l.

Solution. 1. The progression is

TERM 1	TERM 2	TERM 3	TERM 4	TERM 10	TERM n
a	$a + d$	$a + 2d$	$a + 3d$	$\cdots$?	$\cdots$?

2. What is the 10th term? the 18th? the 35th?
3. Similarly, the coefficient of d in the nth term is ——.
4. $\therefore \boxed{l = a + (n - 1)d}$

Example. Find the 10th term of 8, 5, 2, $\cdots$.

Solution. 1. In this A. P., $a = 8$, $d = -3$, $n = 10$, and $l = ?$
2. The formula is $l = a + (n - 1)d$.
3. $\therefore l = 8 + 9(-3) = 8 - 27$, or -19.

214. The terms of an arithmetic progression between any two other terms are the **arithmetic means** of these two terms.

Thus, the *three arithmetic means* of 2 and 14 are 5, 8, 11, since 2, 5, 8, 11, 14 form an arithmetic progression.

A single arithmetic mean of two numbers is particularly important. It is called **the arithmetic mean** of the numbers.

When two numbers are given, *any special number of arithmetic means can be inserted between them.*

Example. Insert five arithmetic means between 13 and -11.

Solution. 1. There will be an arithmetic progression of 7 terms, in which $a = 13$, $l = -11$, and $n = 7$. Find d.
2. $l = a + (n - 1)d$. $\therefore -11 = 13 + 6d$, or $d = -4$.
3. The progression is: 13, 9, 5, 1, -3, -7, -11.

Check. This *is* an A. P. which has five terms between 13 and -11.

EXERCISE 121

Find:

1. The 10th term of 4, 10, 16, ⋯; also the 18th.
2. The 12th term of 17, 14, 11, ⋯; also the 26th.
3. The 16th term of − 3, − 8, − 13, ⋯; also the 36th.
4. The 21st term of 2, 2.05, 2.10, ⋯; also the 46th.
5. The 11th term of 1, $1\frac{1}{2}$, 2, ⋯; also the 52d.
6. What term of the progression 1.05, 1.10, 1.15, ⋯ is 2.25?

Solution. 1. This is an A. P. in which $a = 1.05$, $d = .05$, $n = ?$ and $l = 2.25$. The formula is $l = a + (n - 1)d$.

2. $$\therefore 2.25 = 1.05 + (n - 1) \cdot .05.$$

(Complete this solution.)

7. What term of the progression 7, 4, 1, ⋯ is − 80?
8. What term of the progression − 51, − 43, − 35, ⋯ is 205?
9. What term of the progression $\frac{1}{6}$, $\frac{4}{3}$, $\frac{5}{2}$, ⋯ is $23\frac{1}{2}$?
10. What term of the progression 1.07, 1.14, 1.21, ⋯ is 3.17?

Find the common difference:

11. If the first term of an A. P. is 6 and the 27th term is 188.
12. If the first term is 8 and the 21st term is 108.
13. A man is paying for a house on the installment plan. His payments during the first three months are \$20.00, \$20.10, and \$20.20. What will his 20th and 30th payments be?
14. The first three payments on a washing machine are \$8.00, \$8.04, and \$8.08. What will the 12th payment be?
15. In a Christmas savings fund, the payments to be made for 50 weeks are 5¢, 10¢, 15¢, etc. What will the 40th payment be? the 50th?
16. Another savings fund plan calls for payments for 50 weeks as follows: \$5.00, \$4.90, \$4.80, etc. What will the 25th payment be? the 50th?

EXERCISE 122

1. Insert three arithmetic means between 1 and 17.

2. Insert four arithmetic means between -14 and 16.

3. Insert seven arithmetic means between 8 and 20.

4. Insert five arithmetic means between $1\frac{1}{2}$ and 6.

5. Insert four arithmetic means between $-\frac{5}{3}$ and -5.

6. Find the arithmetic mean of 5 and 13.

7. Find the arithmetic mean of $(y - 4)$ and $(y + 4)$.

8. Find the arithmetic mean of $\sqrt{3}$ and $\sqrt{27}$.

9. Find the arithmetic mean of a and b. From the result, make a rule for finding the arithmetic mean of any two numbers.

NOTE. The arithmetic mean of two numbers is also called their *average*.

10. (*a*) Find the arithmetic mean of 11 and 12.

(*b*) Check your results by finding this arithmetic mean by use of the formula derived in Example 9.

11. Find the common difference if three arithmetic means are inserted between g and h.

12. Find the common difference if r arithmetic means are inserted between 5 and 45.

13. A lot line is 165 feet long. Fencing it, a man wants posts *about* 12 feet apart.

(*a*) Not counting the end posts, how many posts would he place?

(*b*) *Exactly* how far apart should he place them, to have them equally spaced?

14. An orchardist is placing trees in a line which is 485 feet long. He wants the trees about 25 feet apart.

(*a*) How many trees will he place between the two end trees.

(*b*) How far apart should the trees be so that they will be equally spaced in the row?

215. The sum of the first n terms of an arithmetic progression can be found without writing down all the terms.

Given the first term, a, the number of terms, n, and the nth term, l.

Find the sum of these n terms, S.

Solution. 1. $S = a + (a + d) + (a + 2d) \cdots + (l - 2d) + (l - d) + l.$

2. Writing the terms in reverse order,

$$S = l + (l - d) + (l - 2d) \cdots + (a + 2d) + (a + d) + a.$$

3. Adding the equations in Steps 1 and 2,

$$2S = (a + l) + (a + l) + (a + l) \cdots + (a + l) + (a + l) + (a + l).$$

4. Since there were n terms in equations 1 and 2, and since a sum $(a + l)$ results in Step 3 from each term of equation 1,

$$2S = n(a + l).$$

5. $$\therefore \boxed{S = \frac{n}{2}(a + l)}$$

6. By § 213, $l = a + (n - 1)d$. Substituting this value of l in Step 5,

$$S = \frac{n}{2}[a + \{a + (n - 1)d\}].$$

7. $$\therefore \boxed{S = \frac{n}{2}\{2a + (n - 1)d\}}$$

Note. The formulas in Steps 5 and 7 should be memorized.

Example. Find the sum of the first 12 terms of the progression 8, 5, 2, ⋯

Solution. 1. $a = 8;\quad d = -3;\quad n = 12;\quad S = ?$

2. $S = \frac{n}{2}\{2a + (n - 1)d\}. \quad \therefore S = 6\{16 + 11 \cdot (-3)\}.$

3. $\therefore S = 6(-17) = -102.$

EXERCISE 123

Find the sum of:

1. 14 terms of 4, 9, 14.
2. 18 terms of 6, 1, − 4.
3. 15 terms of − 80, − 71, − 62.
4. 12 terms of \$1.08, \$1.16, \$1.24.

Find the sum of the terms of an arithmetic progression if:

5. The number is 16, the first is 3, and the last is 48.

6. The number is 39, the first is 0, and the last is 50.

7. The number is 19, the first is 24, and the last is -48.

8. The number is 6, the first is $-\frac{2}{7}$, and the last is $\frac{5}{14}$.

9. Find the sum of the even numbers from 2 to 50.

10. Find the sum of the numbers 3, 6, 9, $\cdots$ 99.

11. Find the sum of the odd numbers from 1 to 199.

12. A pile of fence posts has 40 in the first layer, 39 in the second, 38 in the third. There are 20 layers. How many posts are there in the pile?

13. In a certain school system, a teacher is paid $950 for her first year's work, and is given an increase of $50 per year each year thereafter. What will be the teacher's total income during ten years of service?

14. It has been learned that if a marble, placed in a groove on an *inclined plane*, passes over a distance D in one second, then in the second second it will pass over the distance $3D$, in the third, over the distance $5D$, etc. Over what distance will it pass in the 10th second? in the tth second?

15. Through what total distance does the marble in Example 14 pass in 5 seconds? in 10 seconds? in t seconds?

16. Experiment has shown that an object will fall during successive seconds the following distances:

1st second, 16.08 ft.;	3d second, 80.40 ft.;
2d second, 48.24 ft.;	4th second, 112.56 ft.

Find the distance through which the object will fall during the 7th second; the tth second.

17. Find the total distance through which the object in Example 16 falls in 5 seconds; in t seconds.

216. In an arithmetic progression, there are five elements, a, d, l, n, S. Two independent formulas connect these elements, the formula for the sum and the formula for the term l. Hence, if any three of the elements are known, the other two may be found.

Example 1. Given $a = -\frac{5}{3}$, $n = 20$, $S = -\frac{5}{3}$; find d and l.

Solution.

1. $S = \frac{n}{2}(a + l)$. $\therefore -\frac{5}{3} = 10\left(-\frac{5}{3} + l\right)$; whence $l = \frac{3}{2}$.

2. $l = a + (n - 1)d$. $\therefore \frac{3}{2} = -\frac{5}{3} + 19 \cdot d$; whence $d = \frac{1}{6}$.

Example 2. Given $a = 7$, $d = 4$, $S = 403$; find n and l.

Solution.

1. $S = \frac{n}{2}\{2a + (n - 1)d\}$. $\therefore 403 = \frac{n}{2}\{14 + (n - 1) \cdot 4\}$.

2. $\therefore 806 = n\{4n + 10\}$; $4n^2 + 10n - 806 = 0$;
$$2n^2 + 5n - 403 = 0.$$

3. $\therefore n = \frac{-5 \pm \sqrt{25 + 3224}}{4} = \frac{-5 \pm \sqrt{3249}}{4} = \frac{-5 \pm 57}{4}$.
$$\therefore n = -\frac{62}{4}, \text{ or } +13.$$

Since n, the number of terms, must be an integer, n must be 13, and not $-\frac{62}{4}$.

4. $\therefore l = 7 + 12 \cdot 4$, or 55.

NOTE. A negative or a fractional value of n must be rejected, together with all other values depending upon it. Why?

Example 3. The 6th term of an arithmetic progression is 10 and the 16th term is 40. Find the 10th term.

Solution. 1. By the formula $l = a + (n - 1)d$:
$$a + 5d = 10.$$
$$a + 15d = 40.$$

2. Solving the system of equations in Step 1, $d = 3$ and $a = -5$.

3. The 10th term: $l = -5 + 9 \cdot 3 = -5 + 27$, or 22.

EXERCISE 124

1. Given $l = 57$, $d = 4$, $n = 17$; find a and S.

2. Given $a = -7$; $l = 123$, $n = 27$; find d and S.

3. Given $a = 3$, $l = 30$, $S = 165$; find n and d.

4. Given $a = 3$, $l = -27$, $d = -2$; find n and S.

5. Given $a = 2$, $n = 16$, $s = 232$; find l and d.

6. Given $n = 19$, $s = -\frac{57}{2}$, $d = \frac{1}{2}$; find a and l.

7. Given $d = 3$, $n = 10$, $S = 126$; find a and l.

8. Given $a = \frac{11}{2}$, $d = -\frac{3}{4}$, $s = -2$; find n and l.

9. Given $l = 60$, $S = 240$, $n = 16$; find a and d.

10. Given $l = 85$, $d = 5$, $S = 750$; find a and n.

11. Given a, n, and S; derive formulas for l and d.

12. Given d, n, and S; derive a formula for a.

13. Given a, l, and n; derive a formula for d.

14. The 6th term of an arithmetic progression is 25 and the 11th term is 5. Find the 30th term.

15. The 5th term of an arithmetic progression is $\frac{7}{15}$, the 21st term is $\frac{11}{3}$, and the last term is $\frac{17}{3}$. Find the number of the terms.

16. The sum of the 2d and the 9th terms of an arithmetic progression is -8; and the sum of the 5th and the 10th terms is $-\frac{8}{3}$. Find the first term.

17. Find four numbers in arithmetic progression such that the sum of the first two shall be -51, and the sum of the last two shall be 9.

18. Find five numbers in arithmetic progression such that the sum of the second, third, and fifth shall be 7; and such that the product of the first and fourth shall be -14.

19. Find three integers in arithmetic progression such that their sum shall be 6, and their product -192.

Geometric Progression

217. A **geometric progression** (G. P.) is a sequence of numbers called **terms**, each of which, after the first, is derived by multiplying the preceding term by a fixed number called the **ratio.** The ratio therefore equals any term divided by the one preceding it.

Thus 2, 6, 18, 54, ⋯ is a geometric progression. The ratio is 3.

Again, 15, -5, $+\frac{5}{3}$, $-\frac{5}{9}$, ⋯ is a G. P. The ratio is $-\frac{1}{3}$. The next two terms are $+\frac{5}{27}$ and $-\frac{5}{81}$.

EXERCISE 125

Are the following geometric progressions?

1. 2, 4, 8, 16 ⋯

2. 64, 32, 16 ⋯

3. 81, 30, 21

4. 3, -6, 12, -24.

Write the first five terms of the G. P. in which:

	5	6	7	8	9
The first term is . .	-3	72	$\frac{1}{4}$	$\frac{1}{3}y$	a
The ratio is	-2	$\frac{1}{4}$	3	$\frac{1}{2}$	r

10. If the ratio is more than $+1$, then the terms become ____(?)____.

218. **The nth term of a geometric progression** can be found without determining all the preceding terms.

Given the first term, a; the ratio, r; and the number of terms, n, of a geometric progression.

Find the nth term, l.

Solution. 1. The progression is $a, ar, ar^2, \cdots$

2. The exponent of r in each term is 1 less than the number of the term. Hence the 8th term would be ar^7 and the 20th term, ar^{19}.

3. Similarly, the nth term must be ar^{n-1}

$$\therefore \boxed{l = ar^{n-1}}$$

EXERCISE 126

1. Find the 8th term of 1, 2, 4, ⋯
2. Find the 6th term of 5, 3, $\frac{9}{5}$, ⋯
3. Find the 11th term of 4, $-$ 12, 36, ⋯
4. Find the 7th term of $-\frac{3}{2}$, 3, $-$ 6, ⋯
5. Find the 9th term of 5, $\frac{5}{2}$, $\frac{5}{4}$, ⋯
6. Find the 10th term of $\frac{m}{128}, \frac{m^2}{64}, \frac{m^3}{32}, \cdots$
7. Indicate the 12th term of 1, $(a + b)$, $(a + b)^2$, ⋯
8. Indicate the 16th term of 1, $\frac{1}{3}$, $\frac{1}{9}$, ⋯; also the tth term.
9. Indicate the 12th term of $x, \frac{x^2}{4}, \frac{x^3}{16}, \frac{x^4}{64}, \cdots$; also the $(n + 1)$th term.
10. What term of the progression 5, 10, 20, 40, ⋯ is 640?
11. What term of the progression 7, 14, 28, ⋯ is 448?
12. What term of the progression 12, 6, 3, ⋯ is $\frac{3}{32}$?
13. If the first term of a geometric progression is 4 and the 5th term is $\frac{4}{81}$, what is the ratio?

Find the ratio of the geometric progression if:

14. The first term is 4 and the 5th term is $\frac{1}{64}$.
15. The first term is $\frac{2}{9}$ and the 6th term is 54.
16. The first term is 64 and the 10th term is $\frac{1}{8}$.
17. The first term is 2 and the 7th term is 1458.

Find by logarithms the value of:

18. The 15th term of the G. P. of which $a = 25$ and $r = 4$.
19. The 20th term of the G. P. of which $a = 1$ and $r = 1.04$.
20. The 30th term of the G. P. of which $a = 50$ and $r = 1.02$.
21. Find r, to hundredths, if $a = 5$, $n = 10$, and $l = 50$.
22. Find r, to hundredths, if $a = 2$, $n = 20$, and $l = 100$.

219. The terms of a geometric progression between any two other terms are called the **geometric means** of those two terms.

Thus, the three geometric means of 2 and 162 are 6, 18, and 54, since 2, 6, 18, 54, 162 form a geometric progression.

A single geometric mean of two numbers is particularly important. It is called **the geometric mean** of the numbers.

When two numbers are given, any specified number of geometric means may be inserted between them.

Example. Insert three *real* geometric means between 9 and $\frac{16}{9}$.

Solution. 1. There results a geometric progression of 5 terms, in which $a = 9$, $l = \frac{16}{9}$, and $n = 5$. Find r.

2. $$l = ar^{n-1}. \quad \therefore \frac{16}{9} = 9 \cdot r^4, \text{ or } r^4 = \frac{16}{81}.$$
$$\therefore r = \sqrt[4]{\tfrac{16}{81}} = \pm \tfrac{2}{3}.$$

3. The progression is: 9, 6, 4, $\frac{8}{3}$, $\frac{16}{9}$, or 9, -6, 4, $-\frac{8}{3}$, $\frac{16}{9}$.

NOTE. The other two fourth roots of $\frac{16}{81}$ are imaginary, and imaginary *means* result from using them.

Check. Each G. P. has three terms between 9 and $\frac{16}{9}$.

EXERCISE 127

1. Insert 4 geometric means between 1 and 243.
2. Insert 5 geometric means between 1 and 64.
3. Insert 2 geometric means between $\frac{1}{16}$ and 4.
4. Find the geometric mean of 9 and 81.
5. Find the geometric mean of $4y$ and $16y^7$.
6. Find the geometric mean of $4n$ and $\frac{1}{16}n$.
7. Find the geometric mean of $\dfrac{a}{b^2}$ and $\dfrac{b^2}{a}$.
8. Find the geometric mean of x and y.
9. Insert 3 real geometric means between 4 and 16.
10. Insert 2 real geometric means between x and y.
11. Insert 3 real geometric means between a and b.
12. Find by logarithms the ratio if 6 geometric means are inserted between 5 and 50.

220. The sum of the first n terms of a geometric progression can be found without writing down the n terms.

Given the first term, a, the ratio, r, and the number of terms, n, of a geometric progression.

Find the sum of the n terms, S.

Solution. 1. $S = a + ar + ar^2 \cdots + ar^{n-2} + ar^{n-1}$

2. M_r $rS = ar + ar^2 \cdots + ar^{n-1} + ar^n$

3. $S - rS = a - ar^n$

$\therefore (1 - r)S = a - ar^n.$

5. $D_{(1-r)}$

$$\boxed{S = \frac{a - ar^n}{1 - r}}$$

6. Since $l = ar^{n-1}$, $rl = ar^n$

$$\therefore \boxed{S = \frac{a - rl}{1 - r}}$$

Example. Find the sum of the first 6 terms of 2, 6, 18, $\cdots$

Solution. 1. $a = 2, r = 3, n = 6$. Find S.

2. $S = \dfrac{a - ar^n}{1 - 3}$. $\therefore S = \dfrac{2 - 2 \cdot 3^6}{1 - 3} = \dfrac{2 - 1458}{-2} = \dfrac{-1456}{-2} = 728.$

EXERCISE 128

Find the sum of the first:

1. Seven terms of the progression 6, 12, 24, $\cdots$
2. Eight terms of the progression 20, 10, 5, $\cdots$
3. Six terms of the progression 4, -12, $+36$, $\cdots$
4. Five terms of the progression $\frac{1}{36}$, $-\frac{1}{12}$, $\frac{1}{4}$, $\cdots$
5. Five terms of the progression -4, 20, -100, $\cdots$
6. Twelve terms of the progression 1, x^2, x^4, x^6, $\cdots$
7. Fourteen terms of the progression 5, $5b^2$, $5b^4$, $\cdots$
8. Find the sum of 20 terms of 1, $(1 + r)$, $(1 + r)^2$, $\cdots$
9. Find the sum of the first 10 powers of 2.

Infinite Geometrical Progressions (*Optional Topic*)

Example 1. Consider the progression $5, \frac{5}{3}, \frac{5}{9}, \cdots$. What happens as the number of terms increases *infinitely?*

Solution. 1. The ratio is $\frac{1}{3}$.

2. When $n = 4$, $l = ar^{n-1} = 5(\frac{1}{3})^3 = \frac{5}{27}$.

$$S_n = \frac{a - rl}{1 - r} = \frac{5 - \frac{1}{3} \cdot \frac{5}{27}}{1 - \frac{1}{3}} = \frac{5 - \frac{5}{81}}{1 - \frac{1}{3}}.$$

3. When $n = 10$, $l = 5(\frac{1}{3})^9 = \dfrac{5}{19{,}683}$.

$$S_n = \frac{a - rl}{1 - r} = \frac{5 - \frac{1}{3} \cdot \frac{5}{19{,}683}}{1 - \frac{1}{3}} = \frac{5 - \frac{5}{59{,}049}}{1 - \frac{1}{3}}.$$

4. As n increases, l decreases; also the term rl of S decreases.

5. If n becomes *infinitely large*, l becomes approximately zero; therefore rl also becomes approximately zero.

6. $\therefore S_n$ is almost $\dfrac{5 - 0}{1 - \frac{1}{3}}$ or $\dfrac{15}{2}$ when n is infinitely large.

Consider now the geometric progression in which the first term is a and the ratio, r, is less than 1.

The sum of n terms is $S_n = \dfrac{a - ar^n}{1 - r}$.

When r is less that 1, r^n decreases as n increases and becomes approximately zero as n becomes infinitely large.

For example, the 30th power of $\frac{1}{2}$ is $\dfrac{1}{1{,}073{,}741{,}824}$ and this is certainly almost zero. Moreover, remember this is *only* the 30th power of $\frac{1}{2}$. And an infinitely large power, — how negligibly small it would be!

No matter what a may be, if it is multiplied by a number which is approximately zero, the product also is approximately zero. Therefore ar^n is approximately zero.

Hence the sum of an infinite number of terms of a geometric progression in which r is numerically less than 1 is given by the formula

$$S = \frac{a}{1 - r}.$$

Example 2. What is the sum of an infinite number of terms of $4, -\frac{8}{3}, \frac{16}{9}, \cdots$?

Solution. 1. $-\frac{8}{3} \div 4 = -\frac{2}{3}$; also $(\frac{16}{9}) \div (-\frac{8}{3}) = -\frac{2}{3}$.

$$\therefore r = -\tfrac{2}{3}.$$

2. $\therefore$ the absolute value of r is $\frac{2}{3}$, which is less than 1. $a = 4$.

3. $$\therefore S = \frac{a}{1-r} = \frac{4}{1+\frac{2}{3}} = \frac{4}{\frac{5}{3}} = \frac{12}{5}, \text{ or } 2.4.$$

Example 3. What is the sum of an infinite number of terms of the progression $1, \frac{1}{2}, \frac{1}{4}, \cdots$?

Solution. 1. $$a = 1; \quad r = \tfrac{1}{2}; \quad S = \frac{a}{1-r}.$$

2. $$\therefore S = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

Check. Does $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$ approximately equal 2? (*Hint.* Think of these numbers as $1''$, $\frac{1}{2}''$, $\frac{1}{4}''$, etc.)

EXERCISE 129

Find the sum to infinity of:

1. $8, 2, \frac{1}{2}, \cdots$

2. $3, 1, \frac{1}{3}, \cdots$

3. $12, 3, \frac{3}{4}, \cdots$

4. $5, .5, .05, \cdots$

5. $1, \frac{1}{10}, \frac{1}{100}, \cdots$

6. $y, \frac{1}{3}y, \frac{1}{9}y \cdots$

7. $m, .1\,m, .01\,m, \cdots$

8. $1, -\frac{1}{3}, +\frac{1}{9}, \cdots$

9. $-4, -\frac{1}{2}, -\frac{1}{16}, \cdots$

10. $\frac{1}{5}, -\frac{1}{8}, +\frac{5}{64}, \cdots$

11. Find the sum of the repeating decimal .8181⋯

Solution. 1. $$.8181 \cdots = \frac{81}{100} + \frac{81}{10{,}000} + \text{etc.}$$

2. $$a = \frac{81}{100}, \text{ and } r = \frac{1}{100}, \; S = \frac{a}{1-r}.$$

3. $\therefore$ The value is $\frac{81}{100} \div \left(1 - \frac{1}{100}\right) = \frac{81}{100} \div \frac{99}{100} = \frac{81}{99} = \frac{9}{11}.$

Find the value of the following repeating decimals:

12. .6666 ⋯

13. .4444 ⋯

14. .636363 ⋯

15. .8333 ⋯

16. .409090 ⋯

17. .3555 ⋯

18. .1111 ⋯

19. .24333 ⋯

20. .52222 ⋯

EXERCISE 130 (*Optional*)

Miscellaneous Examples

1. Find the sum of the first 10 powers of 3.

2. (*a*) Find the sum of 10 terms of 1, 1.04, $(1.04)^2$, $(1.04)^3$,⋯

(*b*) If you have had the chapter on logarithms, find the value of this sum to three decimal places.

3. If a man deposits in a savings bank $5 on the 2nd of January, $10 on the 2nd of February, $20 on the 2nd of March, etc.; how much will he have deposited by the end of the year.

4. A clock strikes at each of the hours, from 1 to 12 inclusive, a number of times corresponding to the hour; and besides, in each hour, it strikes once at the end of the first quarter, twice at the end of the 2nd quarter, and 3 times at the end of the third quarter. How many times does it strike during the twelve hours?

5. (*a*) Find the sum of 25 terms of 1, 1.025, $(1.025)^2$, $(1.025)^3$,⋯

(*b*) By logarithms find the value of this sum.

6. What is the sum of the 7th to the 12th terms, inclusive, of the geometric progression whose first term is 15 and whose ratio is $\sqrt{2}$?

7. On a debt of $4,000, a man was paying 7% interest. At the end of each year, he plans to pay $200 on the principal, and also the interest on the balance of the debt during that year.

(*a*) Write down his payments to be made at the end of years one, two and three.

(*b*) How many years will it take him to pay off the debt?

(*c*) How much will he have paid by the time he has freed himself of the debt?

8. How many poles will there be in a pile of telegraph poles if there are 25 in the first layer, 24 in the second, etc. and 1 in the last?

9. What term of the progression 5, 10, 20, ⋯ is 160.

10. Given $a = \frac{1}{4}$, $l = \frac{35}{4}$, and $S = \frac{315}{2}$ in an A. P.; find d and n.

11. If a man earns $360 during his first year of work, and is given an increase of $50 per year for each succeeding year, what is his salary during his 10th year, and how much has he earned altogether during the 10 years?

12. A father agrees to give his son 5¢ on his fifth birthday, 10¢ on his sixth, and each year up to the 21st inclusive to double the gift of the preceding year. How much will he have given him altogether after his 21st birthday?

13. Each year a man saves half as much again as he saved the preceding year. If he saved $128 the first year, to what sum will his savings amount at the end of seven years?

14. What is the sum of the following fifty payments: 5¢, 10¢, 15¢, ⋯ etc.

15. If at the beginning of each of 10 years a man invests $100 at 6% simple interest, to what does the principal and interest amount at the end of the 10th year?

16. Find the 10th term of the arithmetic progression whose first term is 7 and whose 16th term is 97.

17. Find the arithmetic mean of $\sqrt{2}$ and $\sqrt{18}$.

18. What term of the progression 3, 6, 12, 24, ⋯ is 384?

19. Insert 3 geometric means between 3 and 12.

20. A man has a debt of $3000, upon which he is paying 6% interest. At the end of each year he plans to pay $300 and the interest on the debt, which has accrued during the year. How much interest will he have paid when he has freed himself of the debt?

21. (*a*) What is the arithmetic mean of 3 and 12?
(*b*) What is the geometric mean of 3 and 12?

EXERCISE 131. CHAPTER MASTERY-TEST

1. What sort of progression is:

(*a*) 75, 71, 67, 63, ⋯?

(*b*) 100, 80, 60, 40, ⋯?

2. Derive the formula for the nth term of an arithmetical progression.

3. Derive the formula for the sum of the first n terms of a geometric progression.

4. What is the 10th term of the progression 5, − 10, 20, ⋯?

5. Find the sum of 29 terms of the progression 3, 8, 13, ⋯

6. Insert 11 arithmetic means between − 20 and + 20.

7. If the third term of an arithmetic progression is 15 and the seventh term is 39, what is the first term and the common difference?

8. In a savings club, the successive payments are 10¢, 15¢, 20¢, etc. for 50 weeks. How much is the total of all the payments?

9. Suppose a boy were to save 2¢ during the year when he becomes five years old, 4¢ when he becomes six years old, 8¢ when he becomes seven years old, etc. How much would he have when he becomes 22 years old?

10. What is the sum of the even numbers from 2 to 200 inclusive?

11. Given $l = -21$; $d = -2$; $m = 15$. Find a and S.

12. What is the sum of 15 terms of 1, $(1 + r)$, $(1 + r)^2$, ⋯

13. What is the sum of the ten numbers beginning with 4, and continuing thus: 4, 8, 12, 16, ⋯?

14. What is the sum of the ten numbers beginning with 4 and continuing thus: 4, 16, 64, 256, ⋯?

15. (*a*) What is the sum of ten terms of 1, $\frac{1}{2}$, $\frac{1}{4}$, ⋯?

(*b*) What is the sum of an infinite number of the terms?

EXERCISE 132. CUMULATIVE REVIEW

1. Find the prime factors of each of the following:

(a) $a^2 - x^2 - 2x - 1$

(b) $5a^3 + 9a^3y - 18a^3y^2$

(c) $64x^{12} - y^6$

(d) $6(m-n)^2 - (m-n) - 2$

(e) $2x^2 + 7x - 15$

(f) $a^6 - a^4 + a^2 - 1$

(g) $x^{10} + 2x^5 + 1$

(h) $x^{3n} - 6x^{2n} + 5x^n$

(i) $m^{4x} - 625$

(j) $(x^2 - y^2 - z^2)^2 - 4y^2z^2$

2. Rationalize the denominator of $\dfrac{2 - 3\sqrt{5}}{8 + \sqrt{5}}$

3. Find the value of each of the following expressions:

(a) $(-125)^{-\frac{2}{3}}$ (b) $\dfrac{5^{-2} + 3 \cdot 8^0}{3 + 4 \cdot 5^{-1}}$

4. Determine, without solving the equation, the nature of the roots of each of the following equations, — giving your reason for your decision.

(a) $9x^2 - 3x + .25 = 0.$ (b) $6x^2 - 4x + .5 = 0.$

5. (a) Draw the graph of the equation $y = 3x^2 - 4x.$

(b) From the graph, determine the roots of the following equations, correct to tenths:

(1) $3x^2 - 4x = 0.$ (2) $3x^2 - 4x = 2.$

6. Determine graphically the common solutions of the system

$$\begin{cases} x^2 - y^2 = -9 \\ 2x = y + 1. \end{cases}$$

7. If the $\log x = 2.7153$, what is:

(a) $\log 10x$? (b) $\log x^2$? (c) $\log \sqrt[3]{x}$?

8. Solve and check: $\sqrt{z - 6} + \sqrt{z} = \dfrac{3}{\sqrt{z - 6}}$

9. Solve the system $\begin{cases} 2x^2 - 3xy = -4. \\ 4xy - 5y^2 = 3. \end{cases}$ (Group your answers)

10. Find two numbers whose difference is 4, and such that the sum of their reciprocals is $\frac{3}{8}$.

XV. THE BINOMIAL THEOREM

221. The **binomial theorem** is a formula for expanding any power of a binomial.

By actual multiplication:

$$(a + x)^2 = a^2 + 2\,ax + x^2. \qquad (1)$$

$$(a + x)^3 = a^3 + 3\,a^2x + 3\,ax^2 + x^3. \qquad (2)$$

$$(a + x)^4 = a^4 + 4\,a^3x + 6\,a^2x^2 + 4\,ax^3 + x^4. \qquad (3)$$

Rule. To expand any power of a binomial, like $(a + x)^n$:

1. *The exponent of a in the first term is n and decreases by 1 in each succeeding term until it becomes 1 in the next to the last term.*

2. *The first term does not contain x. The exponent of x in the second term is 1 and increases by 1 in each succeeding term until it becomes n in the last term.*

3. *The coefficient of the first term is 1; of the second is n.*

4. *If the coefficient of any term be multiplied by the exponent of a in that term, and the product be divided by the number of the term, the quotient is the coefficient of the next term.*

Note 1. Observe that the number of the terms is $n + 1$; that is, one more than the exponent of the binomial.

Note 2. Observe also that the sum of the exponents of a and x in each term is n.

Thus in (3) the sum of the exponents in each term is 4.

Note 3. Observe that the coefficients of terms "equidistant" from the ends are the same.

Thus: in (3) the 2nd and 4th terms have coefficient 4.

Note 4. When the second term of the binomial is negative, the terms of the expansion are alternately positive and negative if n is a positive integer.

Note 5. When the terms of the binomial are complicated monomials, place each in parentheses, and afterwards simplify as in Example 2, Page 215.

Example 1. Expand $(a + x)^5$.

Solution. 1. The exponents of a are 5, 4, 3, 2, 1. The exponents of x, starting with 1 in the second term, are 1, 2, 3, 4, and 5. Writing the terms *without the coefficients* gives:

$$a^5 + (?)a^4x + (?)a^3x^2 + (?)a^2x^3 + (?)ax^4 + x^5.$$

2. The coefficient of the first term is 1, and of the second term is 5 (Rule 3). Multiplying 5, the coefficient of the second term, by 4, the exponent of a in the second term, and dividing by 2, the number of the term, gives 10, the coefficient of the third term.

Filling in the coefficients in this manner gives:

$$(a + x)^5 = a^5 + 5\,a^4x + 10\,a^3x^2 + 10\,a^2x^3 + 5\,ax^4 + x^5.$$

Example 2. Expand $\left(2 - \frac{m}{2}\right)^4$. Here "$a$" $= 2$; $x = \left(-\frac{m}{2}\right)$.

$$\therefore \left(2 - \frac{m}{2}\right)^4 = 2^4 + 4(2)^3\left(-\frac{m}{2}\right)$$
$$+ 6(2)^2\left(-\frac{m}{2}\right)^2 + 4(2)\left(-\frac{m}{2}\right)^3 + \left(-\frac{m}{2}\right)^4$$

$$\therefore \left(2 - \frac{m}{2}\right)^4 = 16 - 4 \cdot 8 \cdot \frac{m}{2} + 6 \cdot 4 \cdot \frac{m^2}{4} - 4 \cdot 2 \cdot \frac{m^3}{8} + \frac{m^4}{16}$$

$$\therefore \left(2 - \frac{m}{2}\right)^4 = 16 - 16\,m + 6\,m^2 - m^3 + \frac{m^4}{16}.$$

EXERCISE 133

Expand the following:

1. $(m + n)^5$
2. $(x - y)^4$
3. $(x + 1)^5$
4. $(b - 2)^4$
5. $(r^2 - s^2)^5$
6. $(a - 2\,b)^4$
7. $(2\,b + c)^5$
8. $(1 + y)^8$
9. $(1 - y^2)^6$
10. $(a - 4\,b)^4$
11. $(n - \frac{1}{2})^4$
12. $(\frac{1}{3} + a^2)^6$
13. $(2\,x - 3)^5$
14. $(ax^2 + b)^8$
15. $(4 + x^3)^4$
16. $\left(\frac{x}{a} + \frac{y}{b}\right)^6$
17. $\left(\frac{a}{2} - \frac{b}{3}\right)^7$
18. $\left(\frac{1}{y} - y\right)^8$

Write the first three terms of:

19. $\left(\frac{x}{\sqrt{2}} - 2\right)^6$
20. $(x^{\frac{1}{2}} + y^{\frac{1}{3}})^6$
21. $(x^{\frac{1}{4}} - y^{\frac{1}{2}})^8$

222. Writing any selected term of $(a + x)^n$. *(Optional)*

If you follow the rules stated on page 214, you get:

$$(a + x)^n = a^n + n \cdot a^{n-1} \cdot x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}x^3 + \cdots\cdots$$

Let r represent the number of any term: (such as 2, 3, 4, $\cdots$). You will find that the following statements are correct:

Rule. In the rth term:

1. *The exponent of x is $r - 1$ (one less than the number of the term).*
2. *The exponent of a is $n - (r - 1)$, or $n - r + 1$.*
3. *The denominator of the coefficient is $1 \cdot 2 \cdot 3 \cdots\cdots (r - 1)$, the last factor being the same as the exponent of x.*
4. *The numerator of the coefficient is $n \cdot (n - 1) \cdot (n - 2) \cdots\cdots$* etc., *until there are just as many factors as in the denominator.*

$$\therefore \text{ the } r\text{th term } _{(r \text{ not } 1)} = \frac{n(n-1)(n-2) \cdots\cdots (n-r+2)}{1 \cdot 2 \cdot 3 \cdots\cdots (r-1)} \cdot a^{n-r+1}x^{r-1}$$

Note. You *can* use this formula of course. Preferably use the Rule.

Example. What is the 4th term of $(3a^{\frac{1}{2}} - b)^8$?

Solution. 1. $(3a^{\frac{1}{2}} - b)^8 = \{(3a^{\frac{1}{2}}) + (-b)\}^8$.

2. $\therefore$ the 4th term $= \dfrac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3}(3a^{\frac{1}{2}})^5(-b)^3$ {found by following the Rule.

3. $= 56 \times (243a^{\frac{5}{2}})(-b)^3$, or $-13608a^{\frac{5}{2}}b$.

EXERCISE 134

Find the:

1. 5th term of $(m + n)^6$
2. 4th term of $(a - b)^7$
3. 3rd term of $(m + \frac{1}{2}n)^8$
4. 5th term of $(b^3 - a^2)^7$
5. 4th term of $(2x^2 - y)^5$
6. 6th term of $(3x - 2)^8$
7. 5th term of $(\frac{1}{2}a + 2)^6$
8. 7th term of $(\frac{1}{3}x - 3)^8$
9. Write the middle term of $\left(x + \dfrac{1}{x}\right)^6$.

223. Fractional and negative powers of $(a + x)$. (*Optional.*) The only limitation on the use of the rules of § 221 and § 222 when n is a fraction or a negative number is that a shall be greater than x in numerical value. This is proved in higher mathematics.

Example. Expand $(a + x)^{\frac{2}{3}}$ to four terms.

Solution. 1. $n = \frac{2}{3}$. Use Rule, Page 214.

2. $\therefore (a + x)^{\frac{2}{3}}$

$$= a^{\frac{2}{3}} + \frac{2}{3} a^{\frac{2}{3}-1}x + \frac{\frac{2}{3}(\frac{2}{3} - 1)}{1 \cdot 2} a^{\frac{2}{3}-2}x^2 + \frac{\frac{2}{3}(\frac{2}{3} - 1)(\frac{2}{3} - 2)}{1 \cdot 2 \cdot 3} a^{\frac{2}{3}-3}x^3 + \cdots$$

3. $$= a^{\frac{2}{3}} + \frac{2}{3} a^{-\frac{1}{3}}x + \frac{\frac{2}{3}(-\frac{1}{3})}{2} a^{-\frac{4}{3}}x^2 + \frac{\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})}{1 \cdot 2 \cdot 3} \cdot a^{-\frac{7}{3}}x^3 + \cdots$$

4. $$= a^{\frac{2}{3}} + \frac{2}{3} a^{-\frac{1}{3}}x - \frac{1}{9} a^{-\frac{4}{3}}x^2 + \frac{4}{81} a^{-\frac{7}{3}}x^3 + \cdots.$$

EXERCISE 135

Write the first four terms of:

1. $(a + x)^{\frac{1}{2}}$
2. $(1 + x)^{\frac{1}{3}}$
3. $(1 - x)^{\frac{1}{4}}$
4. $(a + 2)^{\frac{1}{2}}$
5. $(b - 3)^{\frac{1}{4}}$
6. $(c - 4)^{\frac{1}{3}}$
7. $(1 + x)^{-2}$
8. $(1 - x)^{-3}$
9. $(a + x)^{-4}$

224. Extraction of roots by the binomial theorem. (*Optional.*) This is of theoretical rather than practical interest.

Example. Find $\sqrt[3]{25}$ approximately to five decimal places.

Solution. 1. The nearest perfect cube to 25 is 27.

2. $\therefore \sqrt[3]{25} = \sqrt[3]{27 - 2} = \{(3^3) + (-2)\}^{\frac{1}{3}}$

3. $$= (3^3)^{\frac{1}{3}} + \frac{1}{3}(3^3)^{-\frac{2}{3}} \cdot (-2) - \frac{1}{9}(3^3)^{-\frac{5}{3}}(-2)^2 + \frac{5}{81}(3^3)^{-\frac{8}{3}}(-2)^3 + \cdots$$

4. $$= 3 - \frac{2}{3 \cdot 3^2} - \frac{4}{9 \cdot 3^5} - \frac{40}{81 \cdot 3^8}$$

5. $= 3 - .07407 - .00183 - .00008$, or 2.92402.

EXERCISE 136

Find the approximate values to four decimal places of:

1. $\sqrt{17}$
2. $\sqrt{51}$
3. $\sqrt[3]{60}$
4. $\sqrt[3]{75}$
5. $\sqrt[4]{90}$
6. $\sqrt[4]{275}$
7. $\sqrt[5]{35}$
8. $\sqrt[6]{60}$

225. Compound interest. When interest on a sum of money invested for one interest period is added to the principal, and, with it, draws interest during the next interest period, and so on, then the money is said to be invested at compound interest.

Thus, the interest on $1 at 4% for 1 year is $.04. If this is added to $1, making $1.04, and if $1.04 draws interest for one year, the interest during the second year is $.0416, and the amount at the end of two years is $1.04 + .0416 or $1.0816.

This is called the **compound amount**, and $.0816 is called the **compound interest.** The interest was **compounded annually** at 4%.

Problem. Find the compound *amount* at the end of n years if one dollar is invested at rate r compounded annually.

NOTE. "Rate r" means 4%, or 5%, or $3\frac{1}{2}$%, etc.

Solution. 1. The interest during the first year is r, and the amount at the end of year 1 is $1 + r$.

2. The interest for the second year is $r(1 + r)$.

$\therefore$ the amount at the end of year two is $(1 + r) + r(1 + r)$, or $(1 + r)^2$.

3. The interest for the third year is $r(1 + r)^2$.

$\therefore$ the amount at the end of the third year is $(1 + r)^2 + r(1 + r)^2$.

This equals $(1 + r)(1 + r)^2$, or $(1 + r)^3$.

4. Similarly, the compound amount at the end of 4 years is $(1 + r)^4$; at the end of 5 years is $(1 + r)^5$; and at the end of n years is $(1 + r)^n$.

Rule. The compound amount of P dollars invested at rate r, compounded annually for n years is $P\ (1 + r)^n$.

Example 1. What is the compound interest on $250 invested at 4% compounded annually for 10 years?

Solution. 1. In this example, $r = .04$ and $n = 10$.

2. $\therefore$ the compound amount of $1 = $(1 + .04)^{10}$.

3. $\text{Log}\ (1.04)^{10} = 10 \cdot \log 1.04 = 10 \times .0170 = .1700$

4. $\therefore 1.04^{10} =$ (about) $1.48.

5. $\therefore$ the compound amount of $1 for 10 yr. = $1.48

6. $\therefore$ the compound amount of $250 = 250 × $1.48, or $370.00

7. $\therefore$ the compound interest = $370 – $250, or $120.00

That is, *about* $120.00. Actually it is $120.05.

Example 2. Find the compound amount of \$350 invested at 4% compounded semi-annually for 10 years.

Solution. 1. Just accept the following rule: "4% compounded semi-annually" is secured by finding interest at "2% compounded annually for double the time."

$\therefore$ \$350 at 4% compounded semi-annually for 10 years
= 350 at 2% compounded annually for 20 years.

2. $\therefore \quad A = 350(1.02)^{20}$
3. $\therefore \log A = \log 350(1.02)^{20}$
4. $\qquad = \log 350 + 20 \log 1.02$
5. $\therefore \log A = 2.7161$
6. $\therefore \quad A = \$520.00$ (about).

log 350	=	2.5441
20 log 1.02	=	.1720
adding		2.7161

EXERCISE 137

Find the compound amount of:

1. \$500 at 4% compounded annually for 10 years.
2. \$500 at 2% compounded annually for 20 years.
3. \$1000 at 5% compounded annually for 5 years.
4. \$1000 at 3% compounded annually for 10 years.
5. \$750 at 5% compounded annually for 5 years.
6. \$400 at 4% compounded semi-annually for 15 years.
7. \$1500 at 3% compounded semi-annually for 12 years.
8. \$2000 at $4\frac{1}{2}$% compounded annually for 10 years.
9. \$875 at 5% compounded semi-annually for 15 years.
10. \$3200 at 4% compounded annually for 25 years.

NOTE. This section gives a mere introduction to the application of the binomial theorem to one of the most important business applications of algebra.

The *mathematics of investment* and *actuarial mathematics* are founded on the mathematics taught in our chapters on exponents, logarithms, progressions (especially geometric), and the binomial theorem. Any student contemplating a course in commerce in any university will find knowledge of these chapters indispensable.

226. **An exponential equation** is one in which the unknown appears as an exponent; as $10 = 4^x$; or $5.6 = 1.03^{1+x}$.

Example 1. In how many years will \$250 amount to \$1000, at 4% compounded semi-annually?

Solution. 1. Using the formula $A = P(1 + r)^n$,

$A = \$1000;\ P = 250;\ r = .02;\ n = 2x.$

$\therefore 1000 = 250(1.02)^{2x}$, or $4 = 1.02^{2x}$.

3. Taking logarithms: $\log 4 = 2x \log 1.02$

4. $$\therefore x = \frac{\log 4}{2 \log 1.02} = \frac{.6020}{.0172}, \text{ or } 35.$$

EXERCISE 138

Solve the following equations:

1. $50 = 3^x$
2. $125 = 2^y$
3. $275 = 4^c$
4. $300 = 5^{2x}$
5. $2.75 = 1.1^{3x}$
6. $24.5 = 3.2^{4x}$
7. $650 = 10^{1+x}$
8. $87.5 = 2.5^{1-x}$
9. $9.37 = 1.05^{3x}$

10. In how many years will \$300 amount to \$500 at 4% compounded annually?

11. In how many years will \$350 amount to \$1000 at 6% compounded semi-annually?

12. In how many years will \$250 double itself at 4% compounded semi-annually?

13. In how many years will \$400 earn \$200 in compound interest, if it is invested at 5% compounded annually?

14. In how many years will \$100 double itself at 4% compounded annually?

15. In how many years will \$100 double itself at 4% compounded semi-annually?

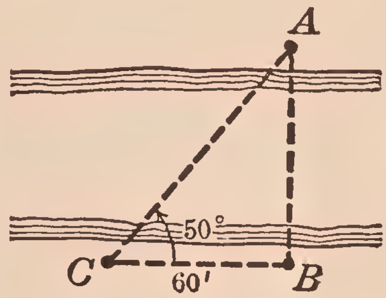
16. In how many years will \$100 double itself at 4% compounded quarterly?

17. In how many years will P dollars double itself at 3% compounded semi-annually?

XVI. TRIGONOMETRY

227. In applied mathematics, distances and angles often cannot be measured directly. In many cases, however, the measures can be secured indirectly by measuring other distances and angles, and making certain computations taught in trigonometry.

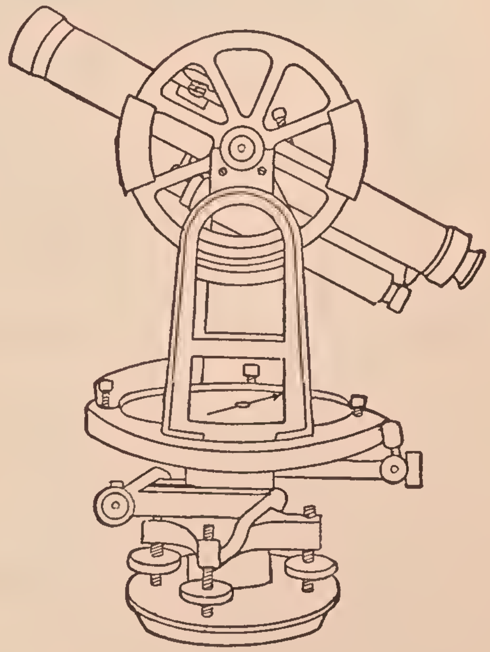
Problem. Find the distance between *A* and *B* on opposite sides of a stream without crossing the stream.



Solution. In trigonometry we learn that the ratio of *AB* to *CB* in a right triangle *ABC* in which angle *C* is 50° is about 1.2.

That is $\frac{AB}{BC} = 1.2$ approximately.

$\therefore AB = 1.2 \times 60$, or $AB =$ about 72 ft.



Observe. *AB* was not measured, it was computed after measuring *CB* and angle *C*.

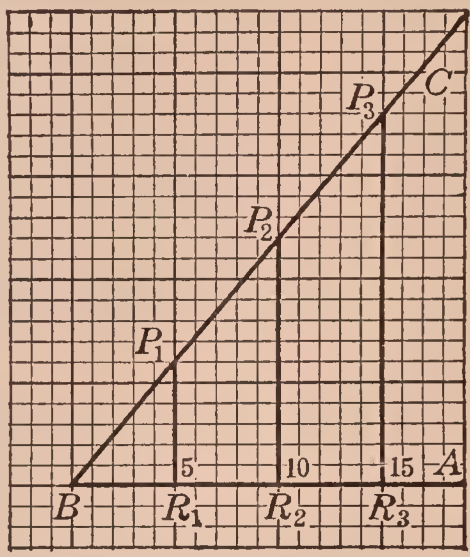
This relatively simple problem illustrates what a powerful tool of computation trigonometry is.

228. Angles are measured by an instrument called the transit. This instrument contains one protractor by which vertical angles are measured and another by which horizontal ones are measured.

Straight lines are measured by a steel tape.

Trigonometry furnishes the means of using the measures.

229. The tangent of an angle. In the adjoining figure, let $\angle ABC = 50°$. Let each space represent 1. P_1R_1 is perpendicular to BA. P_1R_1, P_1B, and BR_1 form triangle BP_1R_1, in which P_1R_1 is the *side opposite* $\angle B$, and BR_1 is the *side adjacent* to $\angle B$.



Observe $\dfrac{P_1R_1}{BR_1} = \dfrac{6}{5} = 1.2.$

Similarly $\dfrac{P_2R_2}{BR_2} = \dfrac{12}{10} = 1.2;$

and $\dfrac{P_3R_3}{BR_3} = \dfrac{18}{15} = 1.2.$

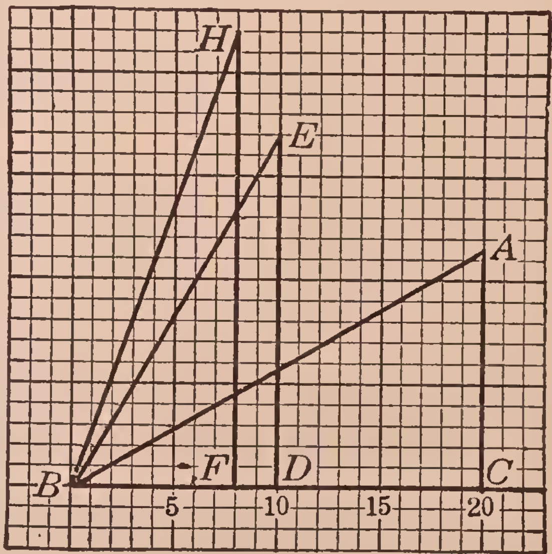
By *similar* triangles, you can prove $\dfrac{PR}{BR} = 1.2$ wherever P is on line BC, when $\angle B = 50°$.

This fixed value is called the **tangent of** $\angle B$ **(tan** B**)**.

$$\text{Tan } B = \frac{PR}{BR} = \frac{\textit{the side opposite } \angle B \textit{ in right } \triangle BPR}{\textit{the side adjacent to } \angle B \textit{ in right } \triangle BPR}.$$

230. When an angle changes, the tangent of the angle changes.

(*a*) In the adjoining figure, let $\angle CBA = 30°$.



Then $\tan \angle CBA = \tan 30°$

$= \dfrac{AC}{BC} = \dfrac{11.5}{20} = .57$, or .6

(*b*) $\angle DBE = 60°$

$\tan 60° = \dfrac{DE}{BD} = \dfrac{17}{10} = 1.7$

(*c*) $\angle FBH = 70°$

$\tan 70° = \frac{22}{8} = 2.75$, or 2.8.

In the table on page 257, the tangents of many angles are given, correct to four decimal places.

Thus: $\tan 30° = .5774$; $\tan 70° = 2.7475$

Observe that **the tangent increases when the angle increases.**

231. The computations in trigonometry *can be done* by ordinary arithmetic, as illustrated below. They can be greatly shortened by using logarithms, if you have studied that topic in Chapter XIII. The College Entrance Examination Board expect that logarithms will be used.

232. Using the table of tangents of angles. (See page 257.)

(*a*) **A problem** *requiring a tangent.*

In the adjoining figure $BC = 25.75$ ft.; angle $B = 63° 24'$; angle $C = 90°$. Find AC.

Solution. 1. $\frac{AC}{BC} = \tan \angle B.$

2. $\therefore AC = BC \tan 63° 24'$

3. $\therefore AC = 25.75 \times 1.9970$

4. $\therefore AC = 51.4227$ or 51.42

$\left.\begin{array}{l}\tan 63° 20' = 1.9912\\ \tan 63° 30' = 2.0057\end{array}\right\}$ diff. = .0145

$4' = .4$ of $10'$

$.4 \times .0145 = .0058$

$\therefore \tan 63° 24' =$
$1.9912 + .0058$, or 1.9970

$\angle CBA$ is the **angle of elevation** of A at B.

$\angle XAB$ is the **angle of depression** of B at A, if XA is parallel to BC, and angle CBA equals angle XAB.

(*b*) **A problem** *in which an angle is to be found.*

In the adjoining figure, find angle ABC.

Solution. 1. Obviously $BA = AD$; and $\therefore BC = CD = 10$ ft.

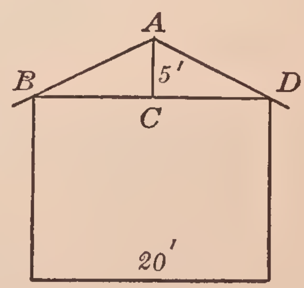
2. $\tan \angle B = \frac{5}{10} = .5000$

$\left.\begin{array}{l}\tan 26° 30' = .4986\\ \tan \angle B \quad\;\, = .5000\end{array}\right\}$ Diff. = .0014 $\Bigg\}$ Tab. diff. = .0036

$\tan 26° 40' = .5022$

$\frac{.0014}{.0036} = \frac{7}{18}.$ $\qquad \frac{7}{18}$ of $10' =$ (almost) $4'$.

3. $\therefore \angle B = 26° 34'.$



EXERCISE 139

1. Find: (*a*) $\tan 27° 15'$; (*b*) $\tan 42° 37'$; (*c*) $\tan 75° 52'$.

2. Find angle x: (*a*) If $\tan x = .4752$; (*b*) If $\tan x = 1.6842$
(*c*) If $\tan x = .8501$; (*d*) If $\tan x = 1.3408$.

233. Using the logarithms of tangents of angles. (See p. 262.)

(a) Solution of problem (a), page 223, by logarithms.

Given $BC = 25.75$; $\angle B = 63° 24'$; $\angle C = 90°$.

Find AC.

Solution. 1. $AC = BC \tan 63° 24'$. (See Steps 1, 2, p. 223.)

2. $\therefore \log AC = \log 25.75 + \log \tan 63° 24'$

$$\left[\begin{array}{rl} \log 25.75 &= 1.4107 \\ \log \tan 63° 24' &= 0.3004 \\ \hline & 1.7111 \end{array}\right]$$

3. $\therefore \log AC = 1.7111$

$$\left[\begin{array}{ll} \log 51.4 = 1.7110 \\ \log AC = 1.7111 \end{array}\right\} \text{Diff.} = 1 \quad \left.\begin{array}{l} \\ \\ \log 51.5 = 1.7118 \end{array}\right\} \text{Tab. diff.} = 8 \quad \tfrac{1}{8} = .125 \Bigg]$$

4. $\therefore AC = 51.4125$, or 51.41.

$\log \tan 63° 20' = 10.2991$
$\log \tan 63° 24' = ?$
$\log \tan 63° 30' = 10.3023$ } Diff. = .0032

$.4$ of $.0032 = .00128$, or $.0013$.

Since the tangent increases when the angle increases,

$\log \tan 63° 24' = 10.2991 + .0013$

or $\log \tan 63° 24' = 10.3004(-10)$

(b) Solution of problem (b), page 223, by logarithms.

Given $AC = 5'$; $BC = 10'$; $\angle C = 90°$.

Find $\angle B$.

Solution. 1. $\text{Tan} \angle B = .5000$

2. $\therefore \log \tan \angle B = \log .5000$

3. $\therefore \log \tan \angle B = 9.6990$

4. $\therefore \angle B = 26° 34'$.

$\log \tan 26° 30' = 9.6977$
$\log \tan \angle B = 9.6990$ } Diff. = .0013
$\log \tan 26° 40' = 9.7009$ } Tab. diff. = .0032

$$\frac{.0013}{.0032} \times 10' = \frac{130'}{32} = 4' \text{ (about)}$$

$$\angle B = 26° 30' + 4'.$$

EXERCISE 140

1. Find the logarithmic tangent of:

(*a*) 27° 30′ (*c*) 40° 42′ (*e*) 70° 18′

(*b*) 25° 35′ (*d*) 60° 56′ (*f*) 15° 33′

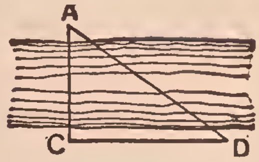
2. Find the angle x if:

(*a*) $\log \tan x = 9.6752$ (*c*) $\log \tan x = 9.7913$

(*b*) $\log \tan x = 9.7044$ (*d*) $\log \tan x = 0.0813$

EXERCISE 141

(Use logarithms or not as your teacher directs.)



1. In the adjoining figure, AC is perpendicular to CD; $CD = 157.5$ ft.; angle $D = 60° 30'$. Find AC.

2. 145.3 ft. from the foot of a high building the angle of elevation of the top is 39° 10′. How high is the building?

3. From the top of a hill known to be 185.0 ft. above the level of the plain, the *angle of depression* (See p. 223) of a house is 22° 20′. How far away is the house from an imaginary point directly below the top of the hill?

4. At a point 175.8 ft. from the foot of a building, the angle of elevation of the top is 51° 30′. How high is the building?

5. From the height of 350.5 ft., the angle of depression of an object on the plain below is 28° 40′. Find the distance of the object from a point in the plain below the point of observation.

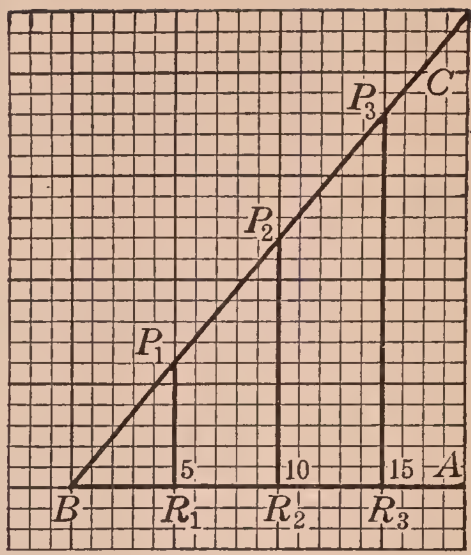
6. The angle of elevation of an airplane at a certain point P is 38° 30′. Point D, 1475 ft. distant, is directly below the airplane. How high is the airplane?

7. At a point 146.8 ft. from the foot of a high chimney, the angle of elevation of its top is 40° 30′. How high is the chimney?

8. From a hilltop 275 ft. above the level of a lake, the angle of depression of one sailboat is 33° 20′. The angle of depression of a second sailboat directly in line with the first boat is 65° 40′. What is the distance between the two boats?

9. An observer in an airplane, which is 1278 ft. high, finds that the angle of depression of a station on the ground is 28° 50′. How far distant is he from a point directly over the station?

10. When the angle of elevation of the sun is known to be 37° 10′, a chimney casts a shadow 125 ft. long. How high is the chimney?



234. The sine of an angle.

If you measure BP_1, P_1R_1, BP_2, P_2R_2, etc., you will find that

$$\frac{P_1R_1}{BP_1} = \frac{P_2R_2}{BP_2} = \frac{P_3R_3}{BP_3}, \text{ etc.}$$

Wherever P is on BC, the ratio of the perpendicular PR to BP always has the same value.

This ratio is called the sine of angle B (**sin** B).

$$\text{Sin } B = \frac{PR}{BP} = \frac{\textit{the side opposite } \angle B \textit{ in rt. triangle } BPR}{\textit{the hypotenuse of rt. triangle } BPR}.$$

A table of the values of the sine for certain angles appears on pages 257 to 261. The *sine increases when the angle increases.*

235. The cosine of an angle. (Refer to the figure above.)

If you measure BR_1, BP_1, BR_2, BP_2, etc., you will find that $\frac{BR_1}{BP_1} = \frac{BR_2}{BP_2} = \frac{BR_3}{BP_3}$. Wherever P is on BC, if $PR \perp BA$, the ratio of BR to BP always has the same value.

This ratio is called the **cosine** of $\angle B$ (**cos** B).

$$\text{Cos } B = \frac{BR}{BP} = \frac{\textit{the side adjacent to } \angle B \textit{ in rt. triangle } BPR}{\textit{the hypotenuse of rt. triangle } BPR}.$$

A table of the values of the cosine for certain angles appears on pages 257 to 261. *The cosine decreases when the angle increases.*

236. Using the tables of sines and cosines.

(*a*) Find sin 43° 52′.

sin 43° 50′ = .6926
sin 44° = .6947
Tab. diff. = 21
2′ = .2 of 10′. .2 × 21 = 4
Since sine increases,
Sin 43° 52′ = .6930

(*b*) Find cos 62° 18′.

cos 62° 10′ = .4669
cos 62° 20′ = .4643
Tab. diff. = 26
8′ = .8 of 10′. .8 × 26 = 21
Since cosine decreases,
Cos 62° 10′ = .4648

237. The logarithms of sines and cosines appear on pages 262 to 266. If they are not used, omit *this page.*

The value of the sine is always more than 0 and less than 1; thus sin 70° = .9397. Therefore the characteristic of the logarithm of a sine must be a negative number. Most of them have characteristic − 1, or 9 − 10. Hence in the table on pages 262 to 266 each logarithm is printed with a characteristic 9 (or 8) and with − 10 understood. Similarly for the logarithms of the cosine.

Example 1. Find the log sin 26° 27′.

Solution.

$$\left.\begin{aligned}\log\sin 26^\circ 20' &= 9.6470\\ \log\sin 26^\circ 27' &= ?\\ \log\sin 26^\circ 30' &= 9.6495\end{aligned}\right\}\begin{aligned}&\text{Tab. diff.}\\ &= 25\end{aligned}\qquad \begin{aligned}7' &= .7 \text{ of } 10'\\ .7\times 25 &= 17.5\\ \text{or } &= 18\end{aligned}$$

Since the sine increases when the angle increases, the

$$\log\sin 26^\circ 27' = 9.6470 + 18, \text{ or } 9.6488.$$

Example 2. Find $\angle x$ if log sin x = 9.2892.*

Solution.

$$\left.\begin{aligned}\log\sin 11^\circ 10' &= 9.2870\\ \log\sin x\quad &= 9.2892\end{aligned}\right\}\begin{aligned}&\text{Diff.}\\ &= 22\end{aligned}\quad \begin{aligned}\log\sin 11^\circ 20' = 9.2934\end{aligned}\quad \text{Tab. diff.} = 64 \qquad \frac{22}{64}\times 10' = 3.7' \text{ or } 4'$$

$\therefore\ \angle x = 11^\circ\, 10' + 4'$, or $11^\circ\, 14'$.

Example 3. Find log cos 47° 43′.

Solution.

$$\left.\begin{aligned}\log\cos 47^\circ 40' &= 9.8283\\ \log\cos 47^\circ 43' &= ?\\ \log\cos 47^\circ 50' &= 9.8269\end{aligned}\right\}\begin{aligned}&\text{Tab.}\\ &\text{diff.}\\ &= 14\end{aligned}\qquad \begin{aligned}.3\times 14 &= 4.2\\ \text{or about } &4.\end{aligned}$$

Since the cosine decreases when the angle increases, subtract the correction.

$$\therefore \log\cos 47^\circ 43' = 9.8283 - 4, \text{ or } 9.8279.$$

Example 4. Find $\angle x$ if log cos x = 9.5253.

Solution.

$$\left.\begin{aligned}\log\cos 70^\circ 20' &= 9.5270\\ \log\cos x\quad &= 9.5253\end{aligned}\right\}\begin{aligned}&\text{Diff.}\\ &= 17\end{aligned}\quad \log\cos 70^\circ 30' = 9.5235 \quad \text{Tab. diff.} = 35 \qquad \tfrac{17}{35} \text{ of } 10' = \text{about } 5'.$$

Since $\angle x$ is between 70° 20′ and 70° 30′, then $\angle x$ = 70° 25′.

* $\angle$ is the symbol for *angle.*

EXERCISE 142

(Use logarithms or not as your teacher directs.)

Omit Examples 1 and 2 if you do not use logarithms.

1. Find: (*a*) log sin 16° 18′ (*b*) log sin 56° 33′
(*c*) log cos 27° 46′ (*d*) log cos 71° 44′

2. Find $\angle x$, if:
(*a*) $\log \sin x = 9.6379$
(*b*) $\log \sin x = 9.9465$
(*c*) $\log \cos x = 9.9271$
(*d*) $\log \cos x = 9.8053$

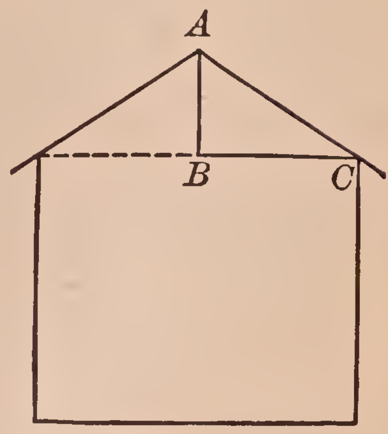


FIG. 1

3. In a triangle XYZ, YZ is 28.50 ft., angle Y is 70° 15′, and XY is 12.75 ft. Draw a triangle to represent these conditions and draw the altitude XW.
(*a*) Find the length of XW.
(*b*) Find the area.

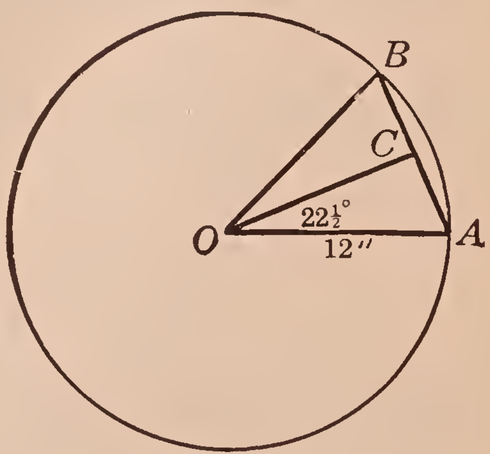


FIG. 2

4. In Figure 1, there is a right triangle in which $AB = 10$ ft., and $BC = 18.5'$.
(*a*) Find angle C. (*b*) Find AC.

5. In Figure 2, OC is perpendicular to AB.
(*a*) Find AC. (*b*) Find OC.

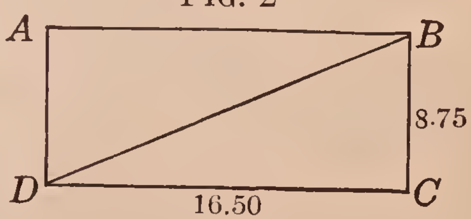


FIG. 3

6. In Figure 3, $ABCD$ is a rectangle.
(*a*) Find $\angle CDB$.
(*b*) Using $\angle CDB$, as found, find DB.

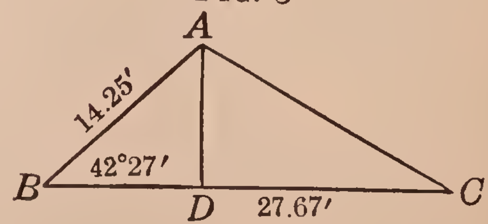


FIG. 4

7. In Figure 4, AD is perpendicular to BC.
(*a*) Find AD.
(*b*) Find the area of $\triangle ABC$.

EXERCISE 143

(Use logarithms or not as your teacher directs.)

1. At a time when the angle of elevation of the sun is known to be $25° 15'$, a chimney casts a shadow 83.25 feet long. How high is the chimney?

2. In rt. $\triangle ABC$, having $\angle A = 90°$, $\angle B = 28° 45'$, and $AB = 27.60$ in., how long are: AC and BC?

3. At a point 145.5 ft. from the foot of a tower, on which stands a flagpole, the angle of elevation of the top of the tower is $35°$, and of the top of the pole $47° 25'$. How high is the tower, and how long is the pole?

4. A ladder 16 ft. long leans against a building. If it makes an angle of $57° 33'$ with the ground, how high on the building does it reach?

5. How long must a guy wire be to reach from the top of a 47.5 ft. pole to a point on the ground 20.75 ft. from the foot of the pole, and what angle will it make with the ground?

6. From a window 35.33 ft. above ground level, the angle of depression of an object on the ground is $41° 18'$. How far is the object from a point on the ground below the observer?

7. Charles, on one side of a stream, holds a 9.8 ft. rod in vertical position, with one end on the ground. Edward, on the other side, measures and finds the angle of elevation of the top of the rod is $16° 45'$. How far apart are the two boys?

8. A hill has a slope of about $13° 50'$. From the foot of the hill to its top is 350.3 ft. How high is the hill?

9. From 500 ft. in the air the angle of depression of a point on the ground is $37° 23'$. How far is the point from directly below the observer?

10. In triangle ABC, $AB = 22.75$ in., $\angle B$ is $42° 33'$, and BC is 24.4 in. Find the area of the triangle.

11. At 119.5 ft. from the foot of a tower, the angle of elevation of the top was $64° 30'$. How high was the tower if the telescope of the transit was 5 ft. above ground level?

12. If the angle of elevation of the sun is $70° 15'$, what is the height of a pole which casts a shadow 17.83 ft. long?

13. In a figure like that for Example 5, page 228, how long is OA, if AC is 7.5 in. and angle AOC is $22\frac{1}{2}°$?

14. Repeat Example 13, if angle AOC is 18°.

15. In a rectangle $RSTW$, $RS = 23.25$ in., and $ST = 14.2$ in.
(*a*) How long is diagonal RT?
(*b*) How large is angle RTS?

16. From the top of a hill which is 83.67 ft. above lake level, the angle of depression of a rowboat on the lake is $18° 30'$. How far (from directly below the observer) is the boat?

17. How long must a ladder be to reach from a point on the ground 17.83 ft. from the side of a building to a point 21.50 ft. up on the side of the building?

18. The height of a tower was observed at two points which were on the same level with and in the same straight line with the foot of the tower. At the nearer point, the angle of elevation of the top of the tower was $45° 30'$ and at the other was $32° 45'$. If the points were 55 ft. 2 in. apart, how high was the tower?

19. In a circle whose radius is 13.5 in., chord AB makes an angle of $38° 55'$ with the radius drawn to B. How long is the perpendicular from the center of the circle to the chord, and how long is the chord if this perpendicular is known to divide the chord into two equal parts?

20. In $\triangle ABC$, $\angle B$ measures 110°; AB is 12.5 in. long and BC is 20.42 in. long. How long is the altitude drawn from A to side BC, and what is the area of $\triangle ABC$?

XVII. VARIATION

238. This chapter contains a summary and slight extension of the subject of **functional dependence** as it has appeared in your courses in mathematics up to this point.

You have learned:

(*a*) That one number is a **function** of one or more other numbers, if a definite value of it corresponds to every value of the other number or numbers.

(*b*) That the number or numbers in an expression or formula which have different values during a discussion are called **variables**; and that those which have one fixed value are called constants.

Thus in $V = \frac{1}{3}\pi r^2 h$, π and 3 are constants, and V, r, or h may be variables.

(*c*) That the dependence of one number upon one or more others can sometimes be expressed algebraically by a formula.

Thus: $A = \frac{1}{2}a(b + c)$ expresses *explicitly* the dependence of A on a, b, and c.

This same formula expresses *implicitly* the dependence of a on A, b, and c. This dependence can be made explicit by solving the formula for a. Then $a = \dfrac{2A}{b + c}$

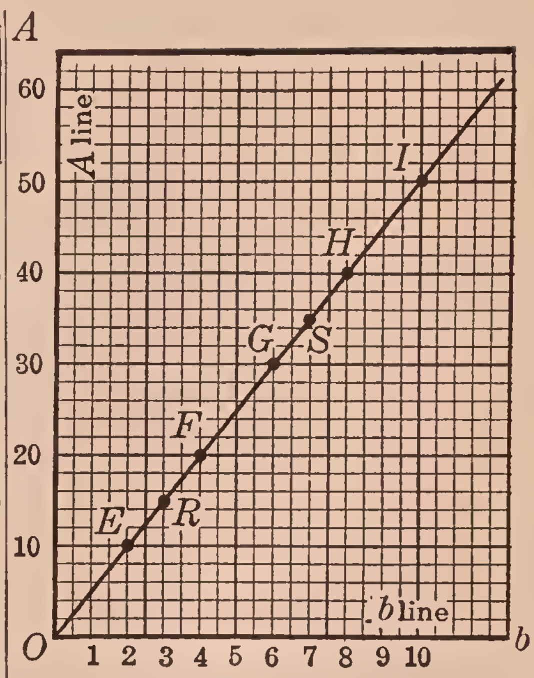
In other words, that a formula expresses a functional relation between the *variables* in it.

(*d*) That every algebraic expression such as x, or $\frac{3}{2}x$, or $2x^2 - 1$, etc. is a function of x.

In particular, that ax or $ax + b$ is a *linear function* of x; that ax^2, or $ax^2 + bx + c$ is a *quadratic function* of x.

239. The graph of the formula $A = 5\,b$.

When $b =$	0	2	4	6	8	10
then $A =$	0	10	20	30	40	50



a. It is clear that:

A increases when b increases;
A decreases when b decreases.

b. When $b = 2,\ A = 10$.
When $b = 4,\ A = 20$.

Observe that A is doubled if b is doubled. Similarly, if b is trebled, then also A is trebled.

c. Observe that $\frac{10}{2} = \frac{20}{4} = \frac{30}{6}$, etc.

That is, the ratio of any value of A to the corresponding value of b is always 5. This fact can be inferred directly from the formula. For, since $A = 5\,b$, then $\frac{A}{b} = 5$.

We say: *A varies directly as b.*

240. While any formula can be studied in this manner, all these facts are true for *any* numbers x and y satisfying an equation of the form $x = my$, or $\frac{x}{y} = m$.

One variable **varies directly** as another if the quotient of any value of the one divided by the corresponding value of the other is constant, as x and y above.

Then such facts as the following are true:

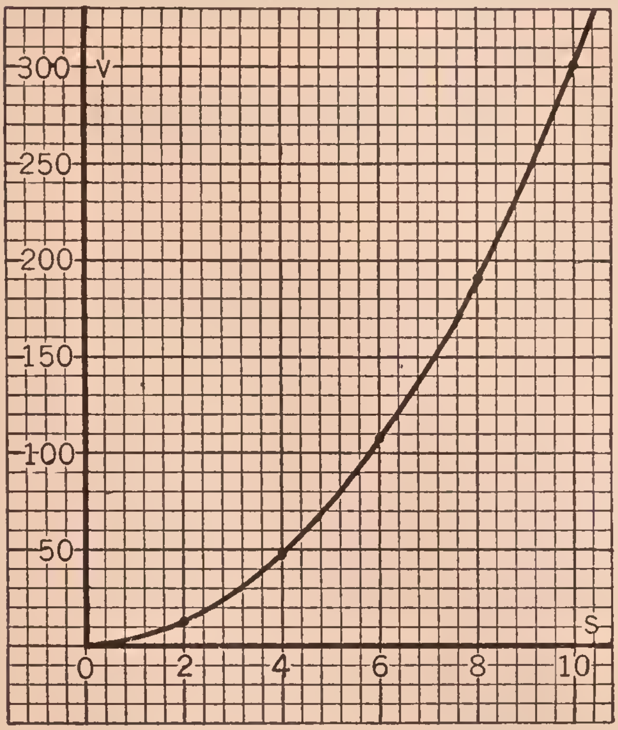
a. x increases when y increases;
b. x decreases when y decreases;
c. x is doubled when y is doubled;
d. x is halved when y is halved; etc.

Example. The perimeter, p, of a square of side s is given by the formula $p = 4\,s$, from which $\frac{p}{s} = 4$. Therefore, p varies directly as s.

241. **A slightly different type of variation.**

Example. *a.* The formula for the volume of the rectangular solid whose base is a square of side s and whose altitude is 3 is $V = 3s^2$.

b. Make a table of corresponding values of s and V and draw the graph representing $V = 3s^2$.



When s =	0	2	4	6	8	10
Then V =	0	12	48	108	192	300

c. Observations.

1. V increases when s increases, but more rapidly than s.
2. When $s = 2$, $V = 12$.
   When $s = 4$, $V = 48$.
   When s is doubled, V is multiplied by 4.
3. $\frac{12}{2^2} = \frac{12}{4} = 3$; $\frac{48}{4^2} = \frac{48}{16} = 3$;
   $\frac{108}{6^2} = \frac{108}{36} = 3$; etc.

Therefore V is proportional to s^2. This fact can be inferred directly from the formula. For, since

$$V = 3s^2, \text{ then } \frac{V}{s^2} = 3.$$

We say, V *varies directly as the square of* s.

In general, one number **varies directly as the square of another** if the quotient of any value of the one divided by the square of the corresponding value of the other is constant.

Example 1. Since the area, A, of a square of side, s, is given by the formula $A = s^2$, then $\frac{A}{s^2} = 1$.

Therefore A varies directly as the square of s.
This means that A is multiplied by 4 when s is doubled.
that A is multiplied by 9 when s is trebled; etc.

242. Another type of variation is illustrated by the formula $I = PRT$. Let $T = 2$. Then $I = 2\ PR$.

Since $I = 2\ PR$, then $\frac{I}{PR} = 2$. Hence, in general I is proportional to the product of P and R.

We say, *I varies jointly* as P and R.

In general, one number **varies jointly** as two or more others when any value of the one divided by the product of the corresponding values of the others is constant.

Example. Since $V = lwh$, $\frac{V}{lwh} = 1$. Hence V varies jointly as l, w, and h.

EXERCISE 144

Complete the following sentences:

1. Since $A = ab$, and therefore $\frac{A}{ab} = ?$, then A varies __________ as a and b. Hence, if a alone is doubled, then A is __________; if b alone is trebled, then A is __________; if a is doubled and b is trebled, then A is __________.

2. Since $A = ab$, if $a = 2$, then $\frac{A}{b} = ?$ Hence A varies __________ as __________. If b is halved, then A is __________. If A is doubled, then b is __________.

3. Since $A = \pi r^2$, then $\frac{A}{r^2} = ?$ Therefore A varies as __________. Hence, if r is doubled, A is multiplied by __________; if r is trebled, A is multiplied by __________.

4. Since $S = 4\ \pi r^2$, then S varies __________ as __________. Hence if r increases, S __________, only __________ rapidly; if r is doubled, then S is __________.

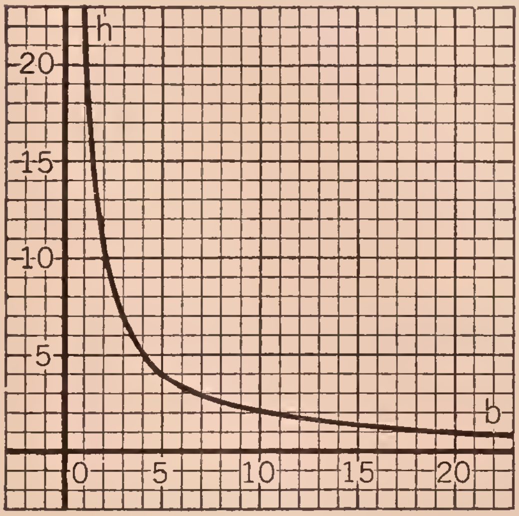
5. Since $V = \frac{4}{3}\pi r^3$, then V varies __________ as __________. Hence if r increases, V __________, only __________ rapidly; if r is doubled, V is multiplied by __________.

243. A quite different type of variation is illustrated by the following example.

Example. *a.* In the formula $A = hb$, for the area of a rectangle, let $A = 20$. Then $hb = 20$ is the formula for all rectangles having an area 20.

b. Make a table of corresponding values of h and b and draw the graph of the relation $hb = 20$.

When $b =$	1	2	4	5	10	20
Then $h =$	20	10	5	4	2	1



c. Observations.

1. When b increases, h decreases.

2. When h increases, b decreases.

3. When $b = 2$, $h = 10$.
   When $b = 20$, $h = 1$.

So, if b is *multiplied* by 10, h is *divided* by 10. Similarly, if b is multiplied by 5, h is divided by 5.

We say h is *inversely proportional* to b, and that b is inversely proportional to h.

In general, one number **varies inversely** as another when the product of any value of the one and the corresponding value of the other is constant.

Example. $p = br$ is a well-known formula. Suppose we consider the case in which p is constant, — like 100, or 50, or 80, etc. Then $br =$ a constant.

Hence b varies inversely as r; that is, the base increases if the rate decreases, and vice versa; if the base is doubled, the rate is halved; etc.

EXERCISE 145

Miscellaneous Examples of Variation

1. If $V = lwh$ and l is constant, then V varies __________ as w and h. If then w alone is doubled, V is __________; if w is doubled and h is trebled, then V is __________.

2. If $V = lwh$ and l and h are constant, then __________.

3. If $I = PRT$, where I, P, R, and T are variable, then I varies __________ as __________.

4. If $I = PRT$, and R and T are constant, then I varies __________ as __________. In this case if I is multiplied by 5, then __________ is __________ by __________.

5. If $I = PRT$, and I and T are constant, then __________ varies __________ as __________. In this case, if P is increased, then __________ is __________.

6. $S = \frac{n}{2}(a + l)$ expresses the dependence of S on __________. If a and l are constant, then S varies __________ as __________.

7. $l = a + (n - 1)d$ expresses the dependence of l on __________. If n and d are constant, l is a function of __________; then if __________ increases, l __________.

8. From the formula in Example 6, $n = \frac{2S}{a + l}$, so n is a function of __________. If S and l are constant, when a increases, n __________. If a and l are constant, when S decreases, n __________.

9. $F = 32 + \frac{9}{5}C$ expresses the functional relation between corresponding Fahrenheit and Centigrade temperature readings. If C increases, then F __________.

10. $Z = 2\pi rh$ is a formula in solid geometry. It indicates that Z varies __________ as __________ and __________. When r and h are both increased, then Z __________. If r is constant, when h is doubled, Z is __________.

ADDITIONAL EXAMPLES FOR REMEDIAL PRACTICE

EXERCISE 146

Removing and inserting symbols of grouping

Remove the parentheses or other symbols of grouping:

1. $3c + (2x - y)$
2. $2x - (y + z)$
3. $4x - (y - z)$
4. $5a - (2b + c)$
5. $7c + (-3x + y)$
6. $5s - (-3t - w)$
7. $6r - (4a - b)$
8. $2a + (3b - c)$
9. $z - (-2x + y)$
10. $6 - (2a + 3c)$

Remove symbols of grouping and combine terms:

11. $3x - (5x - 2)$
12. $(2a - b) - (4a + b)$
13. $(4x - 5) - (3x - 2)$
14. $(2b - 7) + (-3b + 9)$
15. $(-2x + y) - (-2x - y)$
16. $5x^2 - (3x^2 + x) - 4$
17. $(3r - 2s) - (r + 3s)$
18. $(5p + 3q) - (-3p + q)$

Remove all the symbols of grouping, starting with the inner ones:

19. $r + (5r - \{2s - t\})$
20. $x - [y - (2x + y)]$
21. $a - \{3a + (2a - 5)\}$
22. $8 - [-2 + (x + 5)]$
23. $y + \{-3y + (2y - 1)\}$
24. $t + [3 - (2t + 4)]$
25. $w - (5 + [3w - 6])$
26. $s + \{-3s - (2 - s)\}$
27. $9 - [4x - (2x - 6)]$
28. $z + \{-7 - (2z - 3)\}$

Inclose the last three terms of each expression in parentheses, preceded by the sign of the first of the terms which are inclosed:

29. $y^2 + x^2 - 2x + 1$
30. $z^2 - x^2 + 2x - 1$
31. $w^2 - x^2 - 2x - 1$
32. $a^2 + 4 - 4x + 4x^2$
33. $b^2 - 9 - 6c - c^2$
34. $a + bx - cx + dx$
35. $r - sx + tx - wx$
36. $6 - ax - bx + cx$
37. $w - mx + px + qx$
38. $16 - x^2 + 10x - 25$

EXERCISE 147

Factor as taught on page 19.

1. $x^2 + 8x + 15$
2. $m^2 - 11m + 28$
3. $18 - 9t + t^2$
4. $3x^2 + 7x + 2$
5. $6x^2 - 11x + 3$
6. $3x^2 - 19x + 6$
7. $x^2 + 7x + 12$
8. $7y^2 + 9y + 2$
9. $2w^2 - 7w + 3$
10. $24 - 11z + z^2$
11. $6x^2 + 7x + 2$
12. $7r^2 + 4r - 11$
13. $w^2 - 10wr + 24r^2$
14. $5s^2 + 7s - 6$
15. $12x^2 - 5x - 3$
16. $12t^2 - 13t + 3$
17. $c^2 - 11cd + 30d^2$
18. $2x^2 - x - 15$
19. $9y^2 - 6y - 8$
20. $m^2 + 12mn + 35n^2$
21. $7a^2 - 10a + 3$
22. $21A^2 + 2A - 8$
23. $x^2 - 3x - 40$
24. $18x^2 - 3x - 10$
25. $6x^2 + 13x + 6$
26. $y^2 - 2y - 24$
27. $9 - m^2$
28. $x^2 - 10xa + 25a^2$
29. $z^2 - 4z - 21$
30. $x^2 - 4y^2$
31. $32 - 4z - z^2$
32. $1 - 36a^2b^2$
33. $2w^2 - 3w - 20$
34. $x^2 + 12xy + 36y^2$
35. $60 - 4w - w^2$
36. $1 + 14x + 49x^2$
37. $10c^2 + 9c - 9$
38. $4a^2 - 20a + 25$
39. $25a^2 - 1$
40. $54 + 3a - a^2$
41. $9a^2 - 6ab + b^2$
42. $44 + 7t - t^2$
43. $x^2 + 2x - 35$
44. $r^4 + r^2 - 2$
45. $4a^4 - 4a^2b + b^2$
46. $7a^2 - 26ab - 8b^2$
47. $9x^2 - 4y^2$
48. $18 - 7a - a^2$
49. $w^2 - 18w + 72$
50. $4a^2 - 8a - 21$
51. $c^2 - cd - 20d^2$
52. $x^2 - 8xy - 33y^2$

EXERCISE 148

Perform the indicated operations:

1. $\dfrac{4a^2 - 4a + 1}{4a^2 - 1} \cdot \dfrac{2a^2 + a}{8a^3 - 1} \div \dfrac{a^2 - 2a}{a^2 - 4}$

2. $\dfrac{\dfrac{3}{x} + \dfrac{5}{y}}{\dfrac{3}{x} - \dfrac{4}{y}}$

3. $\dfrac{1 - \dfrac{2}{3a}}{a - \dfrac{4}{9a}}$

4. $\dfrac{x - \dfrac{8}{x^2}}{1 - \dfrac{2}{x}}$

5. $\dfrac{a - 1}{a + 1} - \dfrac{a + 1}{a - 1} + \dfrac{4a^2 + 1}{a^2 - 1}$

6. $\dfrac{5x}{x - 3} - \dfrac{4x^2 + 3x - 1}{x^2 + x - 12}$

7. $\left(\dfrac{x + 2}{x} + \dfrac{2}{x - 3}\right)\left(\dfrac{x}{x - 2} - \dfrac{3}{x + 3}\right)$

8. $\dfrac{ax - bx - ay + by}{a^2 - b^2}$

9. $\dfrac{2ac - 2bc - ad + bd}{d^2 - 4c^2}$

10. $\dfrac{a^2 - 2a - 35}{2a^3 - 3a^2} \cdot \dfrac{4a^3 - 9a}{a7 - 49}$

11. $\dfrac{5x + 2}{2x^2 + x - 10} \div \dfrac{1}{x - 2}$

12. $\dfrac{1}{a^2 - 4ab + 4b^2} - \dfrac{1}{4b^2 - a^2}$

13. $x - 3 - \dfrac{x^3 + 27}{x^2 + 3x + 9}$

14. $\dfrac{a}{a^2 + 4a - 60} - \dfrac{a}{a^2 - 4a - 12}$

15. $\dfrac{m - 1}{m - 2} - \dfrac{m + 1}{m + 2} - \dfrac{m - 6}{4 - m^2}$

16. $\left(2x - 1 + \dfrac{6x - 11}{x + 4}\right) \div \left(x + 3 - \dfrac{3x + 17}{x + 4}\right)$

17. Find the value of $\dfrac{x^3 - 1}{x^2 + x + 1}$ when $x = \dfrac{1}{2}$.

18. $\left(2 - \dfrac{x^2 + 4x - 21}{x^2 + 2x - 8}\right) \div \left(\dfrac{x + 1}{x - 2} + \dfrac{x - 3}{x + 4}\right)$

19. $\dfrac{x^2 + 2xy + y^2 - z^2}{x^2 - 2xz + z^2 - y^2}$

20. $\dfrac{ax + ay - bx - by}{a^2 - 2ab + b^2}$

21. $\dfrac{6a^2 - a - 2}{4a^2 - 16a + 15} \cdot \dfrac{8a^2 - 18a - 5}{12a^2 - 5a - 2} \div \dfrac{4a^2 + 6a + 2}{4a^2 - 9}$

EXERCISE 149

Uniform Motion Problems

1. A freight train runs 6 miles an hour less than a passenger train. It runs 80 miles in the same time that the passenger train runs 112 miles. Find the rate of each.

2. A man walked 10 miles. He returned in a car at the rate of 30 miles an hour. If he was gone 4 hours, what was his rate while walking?

3. One auto, traveling at the average rate of 42 miles an hour, and a second, traveling at the average rate of 30 miles an hour, left points which are 252 miles apart and traveled toward each other. In how many hours will they meet?

4. A railroad and an auto highway run beside each other for a considerable distance. If a train left a starting point at 8:00, running 30 miles an hour, and an automobile left the same point at 8:15, running 40 miles an hour, how far from the starting point will the automobile overtake the train?

5. A man traveled 100 miles at the rate of 30 miles an hour. At what rate must he return if he wishes to cut the time for the return trip to three fourths of that for the outward trip?

6. (*a*) One train travels f miles an hour and a second travels s miles an hour. If they start toward each other from points which are d miles apart, leaving at the same time, in how many hours will they meet?

(*b*) After you find the result for part (*a*), answer the following questions:

1. The time is a function of __________?

2. If f and s are constant, then the time increases when the distance d __________.

3. If d and f are constant, and s increases, then the time __________.

4. If d is constant and f and s both increase, then __________.

EXERCISE 150

Mixture Problems

1. One bar of silver alloy is 40% silver; another bar is 25% silver. How many ounces of each must be taken to make 80 oz. of a new alloy (when melted together), which will contain 35% silver?

2. How many pounds of 60¢ candy and of 30¢ candy must be taken to make 50 lb. of mixed candy to sell at 40¢ per pound?

3. Chemically pure sulphuric acid contains 95% of sulphuric acid. How many pints of it and of distilled water must be taken to make a one gallon mixture which shall be 60% pure?

4. "Glycerine and rose water" is a mixture of pure glycerine and perfumed distilled water. How much water must be added to a pint of such a mixture which contains 50% of glycerine to make a new mixture which will contain 40% of glycerine?

5. In a mixture of sand and cement containing one cubic yard, 40% is cement. How much cement must be added so that the resulting mixture will be 50% cement?

6. Stronger ammonia water contains 25% of ammonia. How much water must be added to one gallon of the stronger ammonia water to make ammonia water which is 10% pure?

7. Chemically pure hydrochloric acid contains 33% of hydrochloric acid. How many pints of it and of distilled water are needed to make one quart of dilute hydrochloric acid which is 10% pure?

8. Tincture of iodine contains 6.5% of iodine. How much alcohol must be added to one pint of the tincture of iodine to make a solution which contains 3% of iodine?

9. A normal salt solution contains 85% of salt. How much distilled water must be added to one quart of a normal salt solution to make a solution which contains 5% of salt?

EXERCISE 151

1. A gallon occupies 231 cubic inches of space. Express as a formula the number of gallons, G, which can be held by a tank l feet long, w feet wide, and h feet high.

2. A cord of wood contains 128 cubic feet. Express as a formula the number of cords, C, in a pile of wood m feet long, n feet wide, and r feet high.

3. A bushel contains 2150.4 cubic inches. Express as a formula the number of bushels, B, in a bin a feet long, b feet wide, and c feet high.

4. *a.* One cubic foot contains about $7\frac{1}{2}$ gallons. Express as a formula the number of gallons, G, in a tank r feet long, s feet wide, and t feet high.

b. By the formula of part a, find the number of gallons in a tank 7 feet long, 3 feet wide, and 2 feet high.

c. By the same formula, find the number of gallons in a tank 3 yards long, 1.5 yards wide, and 30 inches high.

5. A bushel of potatoes occupies 1.5 cubic feet. Express by a formula the number of bushels, B, in a bin x feet long, y feet wide, and 3 feet high.

6. *a.* A cubic foot of water weighs 62.5 pounds. Express by a formula the weight in pounds, P, of the water held by a tank L feet long, W feet wide, and H feet high.

b. Express the weight *in tons*, T, of the water in the same tank.

c. Find the number of tons of water in a tank 8 feet long, 4 feet wide, and 3 feet high.

7. *a.* Coal is 1.3 times as heavy as water. Express by a formula the weight in tons, T, of the coal that fills a box l feet long, w feet wide, and h feet high.

b. Find the weight in tons of the coal in a box 9 feet long, 3 feet wide, and 2 feet high.

EXERCISE 152

1. (*a*) Solve $T = \frac{1}{a} + t$ for t.

(*b*) Find t when $T = 413.2$ and $a = 273$.

2. (*a*) Solve $u = \left(\frac{m}{M + m}\right) v$ for M.

(*b*) Find M when $u = 40$; $m = 5.5$; and $v = 1000$.

3. (*a*) $C = \frac{Kab}{b - a}\cdot$ Solve it for b.

(*b*) Find b when $C = 12{,}750$; $K = 315$, and $a = 8$.

4. $s = \frac{c - b}{a - b}\cdot$ (*a*) Solve it for c.

(*b*) Find c when $s = 3.25$, $a = 25$, and $b = 22$.

5. $l = \frac{L - M}{Mt}\cdot$ (*a*) Solve it for t.

(*b*) Find t when $l = .00002$, $M = 90$, and $L = 90.2$.

6. $p = \frac{ad^2}{t} + d$. (*a*) Solve it for a.

(*b*) Find a when $p = 135$, $d = \frac{1}{2}$, and $t = \frac{1}{4}$.

7. $A = P + PRT$. (*a*) Solve it for T.

(*b*) Find T when $A = \$3750$, $P = \$3000$, and $R = 5\%$.

8. $S = \frac{n}{2}(a + l)$. (*a*) Solve it for n.

(*b*) Find n when $S = 473$, $a = 17$, and $l = 69$.

(*c*) Find l when $S = 3275$, $a = 30$, and $n = 10$.

(*d*) Solve for a; find a when $S = 280$, $l = -50$, and $n = 14$.

9. $S = \frac{rl - a}{r - 1}\cdot$ (*a*) Solve it for r.

(*b*) Find r when $S = 2042$, $l = 1024$, and $a = 6$.

(*c*) Solve the same formula for l.

(*d*) Find l when $S = 2400$, $r = 3$, and $a = 6$.

(*e*) Solve the same formula for a.

(*f*) Find a when $S = -200$, $r = \frac{1}{2}$; and $l = 10$.

EXERCISE 153

Solve the following systems for x and y:

1. $\begin{cases} 3x + ay = 3 \\ 2x - by = 4 \end{cases}$

2. $\begin{cases} 4x - my = n \\ x + py = q \end{cases}$

3. $\begin{cases} 3ax - y = b \\ ax + 2y = c \end{cases}$

4. $\begin{cases} mx + ny = 1 \\ cx - 2y = 2 \end{cases}$

5. $\begin{cases} ax - by = 3 \\ bx - ay = 1 \end{cases}$

6. $\begin{cases} cx - dy = a \\ x - by = c \end{cases}$

7. $\begin{cases} mx + ny = m \\ x + my = n \end{cases}$

8. $\begin{cases} 4x - sy = t \\ sx - ry = t \end{cases}$

9. $\begin{cases} 3x - 4y = 5 \\ x + 2y = 6 \end{cases}$

10. $\begin{cases} 5x - 3y = 1 \\ 2x + 5y = 19 \end{cases}$

11. $\begin{cases} 4x + 3y = 1 \\ 3x - 5y = -.7 \end{cases}$

12. $\begin{cases} 5x - 2y = 1.2 \\ 3x + 4y = .2 \end{cases}$

13. $\begin{cases} ax - by = a \\ bx + ay = b \end{cases}$

14. $\begin{cases} ax - by = c \\ bx - ay = c \end{cases}$

15. $\begin{cases} mx + ny = d \\ nx - my = d \end{cases}$

16. $\begin{cases} \frac{2x}{a} + \frac{y}{b} = \frac{2}{3} \\ \frac{3x}{a} - \frac{y}{b} = \frac{1}{6} \end{cases}$

17. $\begin{cases} \frac{4x}{a} - \frac{3y}{b} = \frac{3}{2} \\ \frac{5x}{a} + \frac{y}{b} = \frac{8}{3} \end{cases}$

18. $\begin{cases} \frac{x}{m} - \frac{6y}{n} = -1 \\ \frac{2x}{m} - \frac{7y}{n} = -2 \end{cases}$

19. $\begin{cases} \frac{6x}{c} + \frac{2y}{d} = \frac{5}{2} \\ \frac{5x}{c} - \frac{3y}{d} = \frac{11}{12} \end{cases}$

20. $\begin{cases} \frac{3x}{t} - \frac{y}{r} = 0 \\ \frac{x}{t} + \frac{2y}{4} = \frac{7}{6} \end{cases}$

21. $\begin{cases} \frac{10}{x} - \frac{9}{y} = 2 \\ \frac{8}{x} - \frac{15}{y} = -1 \end{cases}$

22. $\begin{cases} \frac{9}{x} + \frac{10}{y} = 5 \\ \frac{6}{x} + \frac{15}{y} = 5 \end{cases}$

23. $\begin{cases} \frac{6}{x} + \frac{4}{y} = 6 \\ \frac{9}{x} + \frac{5}{y} = 8 \end{cases}$

24. $\begin{cases} 2x + \frac{5}{y} = -9 \\ 4x - \frac{3}{y} = -5 \end{cases}$

25. $\begin{cases} \frac{3}{x} - \frac{1}{y} = -1 \\ \frac{4}{x} + \frac{3}{y} = 7\frac{1}{3} \end{cases}$

26. $\begin{cases} \frac{x}{2} + \frac{2y}{3} = \frac{7}{6} \\ \frac{x}{3} - \frac{5y}{6} = 1 \end{cases}$

27. $\begin{cases} \frac{x}{7} - \frac{3y}{14} = -2 \\ \frac{x}{2} + \frac{7y}{6} = 8 \end{cases}$

28. $\begin{cases} \frac{2}{x} - \frac{3}{y} = -a \\ \frac{5}{x} - \frac{6}{y} = \frac{b}{2} \end{cases}$

29. $\begin{cases} \frac{5x}{a} - \frac{3y}{b} = 11 \\ \frac{7x}{a} + \frac{4y}{b} = -1 \end{cases}$

30. $\begin{cases} \frac{x - c}{d} = \frac{d - y}{2c} \\ \frac{x - y}{c} = 3 \end{cases}$

EXERCISE 154

1. Solve the equation $\sqrt{x-2}+\sqrt{2x+5}=3$.

Solution. 1. $\sqrt{x-2}+\sqrt{2x+5}=3$.
2. $\therefore \sqrt{2x+5}=3-\sqrt{x-2}$.
3. Squaring, $2x+5=9-6\sqrt{x-2}+x-2$.
(Complete as in Examples 1–6.)

NOTE. You will get two *apparent roots*, but one will not check. Remember that the radical indicates the *principal root.*

2. $\sqrt{y+5}-\sqrt{y}=1$

3. $\sqrt{s-3}-1=\sqrt{s-10}$

4. $\sqrt{3m-11}+\sqrt{3m+10}=7$

5. $\sqrt{y+7}-\sqrt{y-5}=2$

6. $\sqrt{x+15}-\sqrt{x-24}=13$

7. $\sqrt{c+20}-\sqrt{c-1}=3$

8. $\sqrt[3]{4y-40}+14=10$

9. $2\sqrt{3x-2}-3\sqrt{x-2}=2$

10. Solve the equation $\sqrt{y+3}-\sqrt{y+8}=-\sqrt{y}$.

Solution. 1. Squaring both members:
$y+3-2\sqrt{(y+3)(y+8)}+y+8=y$.
(Complete the solution as in Examples 1–6.)

11. $\sqrt{x+2}-\sqrt{x-2}=\sqrt{2x}$

12. $\sqrt{2t+1}=2\sqrt{t}-\sqrt{t-3}$

13. $\sqrt{x+5}-\sqrt{8-x}=\sqrt{9-2x}$

14. $\sqrt{3-2c}-\sqrt{7+6c}=\sqrt{1+2c}$

15. $\sqrt{m+3}+\sqrt{3}=\dfrac{6}{\sqrt{m+3}}$

16. $\dfrac{\sqrt{x+2}}{\sqrt{x}}-\dfrac{\sqrt{x}}{\sqrt{x+2}}=\dfrac{2}{3\sqrt{x+2}}$

17. $\dfrac{9+\sqrt{x}}{9-2\sqrt{x}}=\dfrac{11}{5}$

18. $\dfrac{\sqrt{3x+1}-\sqrt{8x}}{\sqrt{3x+1}+\sqrt{8x}}=-\dfrac{3}{13}$

EXERCISE 155

Solve by completing the square:

Example. Solve for x the equation $x^2 + 3\,ax + 4\,b = 0.$

Solution. 1. $$x^2 + 3\,ax = -4\,b.$$

2. $\frac{1}{2}$ of $3\,a = \left(\frac{3\,a}{2}\right).$ $\left(\frac{3\,a}{2}\right)^2 = \frac{9\,a^2}{4}.$

3. $\text{A}\left(\frac{9\,a^2}{4}\right)$ $$x^2 + 3\,ax + \frac{9\,a^2}{4} = \frac{9\,a^2}{4} - 4\,b.$$

4. $$\left(x + \frac{3\,a}{2}\right)^2 = \frac{9\,a^2 - 16\,b}{4}.$$

5. $$\therefore x + \frac{3\,a}{2} = \pm\frac{\sqrt{9\,a^2 - 16\,b}}{2}.$$

6. $$x = \frac{-3\,a}{2} \pm \frac{\sqrt{9\,a^2 - 16\,b}}{2}, \text{ or } \frac{-3\,a \pm \sqrt{9\,a^2 - 16\,b}}{2}.$$

7. $$x_1 = \frac{-3\,a + \sqrt{9\,a^2 - 16\,b}}{2};\quad x_2 = \frac{-3\,a - \sqrt{9\,a^2 - 16\,b}}{2}.$$

Check: Does

$$\left(\frac{-3\,a + \sqrt{9\,a^2 - 16\,b}}{2} + \frac{-3\,a - \sqrt{9\,a^2 - 16\,b}}{2}\right) = -3\,a? \text{ Yes.}$$

Does $$\left(\frac{-3\,a + \sqrt{9\,a^2 - 16\,b}}{2}\right)\left(\frac{-3\,a - \sqrt{9\,a^2 - 16\,b}}{2}\right) = +4\,b?$$

Does $$\left(\frac{(-3\,a)^2 - (9\,a^2 - 16\,b)}{4}\right) = +4\,b?$$

Does $$\frac{9\,a^2 - 9\,a^2 + 16\,b}{4} = 4\,b? \text{ Yes.}$$

1. $x^2 + 2\,ax + c = 0$
2. $x^2 + 4\,bx + t = 0$
3. $x^2 - 6\,bx - c = 0$
4. $2\,x^2 - 4\,x + m = 0$
5. $2\,x^2 - 2\,x - n = 0$
6. $3\,x^2 - 6\,ax + b = 0$
7. $x^2 + 2\,ax - b = 0$
8. $x^2 - 2\,mx = 1 + 2\,m$
9. $x^2 + 6\,x = t^2 - 6\,t$
10. $x^2 - 2\,ax = b^2 + 2\,ab$
11. $ax^2 - 2\,x + 1 = 0$
12. $ax^2 + x + 1 = 0$
13. $ax^2 - 2\,dx + m = 0$
14. $Ax^2 + Bx + C = 0$

EXERCISE 156. QUADRATIC EQUATIONS

Solve these equations by factoring when that is possible; otherwise complete the square or use the formula.

1. $\dfrac{3}{4x} + \dfrac{4x}{3} = -\dfrac{13}{6}$

2. $\dfrac{6}{7-y} + \dfrac{4}{y} = -\dfrac{4}{3}$

3. $\dfrac{2}{x-2} - \dfrac{7}{2x-6} = \dfrac{1}{15}$

4. $\dfrac{3y}{4-5y} + \dfrac{5y-4}{3y} = -\dfrac{8}{3}$

5. $\dfrac{6-x}{x} = 2 - \dfrac{x}{x+4}$

6. $\dfrac{a}{a-6} - \dfrac{2}{a-5} = \dfrac{2a-9}{a-6}$

7. $\dfrac{3}{x-1} = \dfrac{5}{6} + \dfrac{2}{x}$

8. $\dfrac{1}{2} = \dfrac{2}{t-2} + \dfrac{7+3t}{4}$

9. $\dfrac{x}{x-1} = \dfrac{x^2+x-1}{x^2-x} - \dfrac{x-1}{x}$

10. $\dfrac{x}{x+2} - \dfrac{x^2+2x-2}{x^2+5x+6} = \dfrac{x}{x+3}$

11. $\dfrac{2y+1}{7-y} - \dfrac{y-9}{1-3y} = 1$

12. $\dfrac{1}{x-2} + \dfrac{7x}{x+2} = \dfrac{26}{x^2-4}$

13. $\dfrac{3}{4x-3} + 4 = \dfrac{3-2x}{4x}$

14. $\dfrac{2y}{5} = \dfrac{7}{5} - \dfrac{y-6}{y-4}$

15. $\dfrac{5}{3-r} = \dfrac{12}{6-r} + \dfrac{r}{3-r}$

16. $\dfrac{8}{a-3} - \dfrac{a}{a-4} = \dfrac{a-6}{a-4}$

17. $\dfrac{4}{7-x} + \dfrac{5}{x} + \dfrac{9}{2} = 0$

18. $\dfrac{2c}{c-2} - \dfrac{c}{12} = 2$

19. $\dfrac{W}{W-1} - \dfrac{3}{2} = \dfrac{5W}{12}$

20. $\dfrac{x-1}{x} = \dfrac{x}{x+4} - \dfrac{2x-5}{x}$

21. $\dfrac{y}{y-2} = \dfrac{y}{y+2} + \dfrac{4}{3}$

22. $\dfrac{x-3}{x-2} = \dfrac{13}{4} + \dfrac{4}{x}$

23. $\dfrac{5}{5-x} = \dfrac{3x-16}{x-8}$

24. $\dfrac{6m}{7-m} + 5 = \dfrac{3-4m}{3}$

EXERCISE 157

Solve for x, using any convenient method:

1. $x^2 - 4cx - 5c^2 = 0$
2. $3x^2 - rx - 2r^2 = 0$
3. $x^2 + 2xp + c^2 = 0$
4. $3x^2 + 2mx - n = 0$
5. $s = ax + \frac{1}{2}gx^2$
6. $x^2 - Px + (P - 1) = 0$
7. $x^2 + (a + 1)x + a = 0$
8. $x^2 - (a - b)x - ab = 0$
9. $cx^2 - (c + d)x + d = 0$
10. $\pi x^2 + \pi xl - s = 0$

11. (*a*) The square of a certain number equals the sum of that number and n. What is the number?

(*b*) Using your results as formulas, find the number, when $n = 12$; also when $n = 10$.

12. (*a*) If the square of a certain number be increased by n times the number, the result is m. What is the number?

(*b*) Using your results as formulas, what is the number when $n = 8$ and $m = 9$? also when $n = 5$ and $m = 10$?

13. (*a*) There are two consecutive even integers whose product is p. What are they?

(*b*) Using your results as formulas, find the integers when p is 48; also when $p = 80$.

(*c*) Suppose $p = 9$. Do the results have real meaning according to the statement of the problem?

14. (*a*) The length of a certain rectangle exceeds its width by r feet. Its area is s square feet. What are its dimensions?

(*b*) Using your results as formulas, find the dimensions when $r = 12$ and $s = 13$.

(*c*) Also find the dimensions when $r = 10$ and $s = 100$.

15. (*a*) The sum of the base and altitude of a certain rectangle is p feet, and the area of it is A square feet. What are its dimensions?

(*b*) What are the dimensions when $p = 40$ and $A = 375$?

EXERCISE 158

Find:

1. m so that the roots of $x^2 - 14\,mx + 9 = 0$ shall be equal.

2. c so that the roots of $y^2 - 5\,y = 8\,c$ shall be equal.

3. a so that the roots of $ax^2 - 3\,x + 4 = 0$ shall be equal.

4. Determine n so that $2 - \sqrt{3}$ shall be a root of the equation $x^2 + 5\,x + n = 0$.

5. Determine p so that $\sqrt{2} - 1$ shall be a root of the equation $2\,x^2 - px + 4 = 0$.

6. Without solving the equation or substituting in it, determine whether $1 + \sqrt{2}$ and $1 - \sqrt{2}$ are the roots of the equation $x^2 - 2\,x - 1 = 0$.

7. Without solving or substituting in the equation, determine whether $3 - 2\,i\sqrt{2}$ and $3 + 2\,i\sqrt{2}$ are the roots of the equation $x^2 - 6\,x + 17 = 0$.

8. Find the value of m so that the sum of the roots of the equation $3\,mx^2 + (8\,m - 1)x + 7 = 0$ shall be 3.

9. Find the value of c in the equation $2\,x^2 + (2\,c - 1)x + 10\,c^2 = 0$ so that the product of the roots shall be 20.

10. Find the value of k so that the roots shall be equal in the equation $kx^2 + (k - 1)x - 1 = 0$.

11. Find the value of t so that the roots shall be equal in the equation $tx^2 + (2\,t - 1)x - 2 = 0$.

12. Find the value of t so that one root shall be double the other in the equation $tx^2 + (2\,t - 1)x - 2 = 0$.

13. What relation must exist between k and c in order that the roots of $kx^2 - 3\,x + c = 0$ shall be equal?

14. In the equation $x^2 + 2\,kx + 3 = 0$, what must the value of k be so that one root shall be 2 more than the other?

15. What is the value of k in $x^2 + (k^2 - 4)x - 5 = 0$ if the roots are equal in value but opposite in sign?

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* The numbers refer to pages.

TABLE I. CERTAIN POWERS AND ROOTS

No.	Sqs.	Sq. Roots	Cubes	Cube Roots	No.	Sqs.	Sq. Roots	Cubes	Cube Roots
1	1	1.000	1	1.000	**51**	2,601	7.141	132,651	3.708
2	4	1.414	8	1.259	**52**	2,704	7.211	140,608	3.732
3	9	1.732	27	1.442	**53**	2,809	7.280	148,877	3.756
4	16	2.000	64	1.587	**54**	2,916	7.348	157,464	3.779
5	25	2.236	125	1.709	**55**	3,025	7.416	166,375	3.802
6	36	2.449	216	1.817	**56**	3,136	7.483	175,616	3.825
7	49	2.645	343	1.912	**57**	3,249	7.549	185,193	3.848
8	64	2.828	512	2.000	**58**	3,364	7.615	195,112	3.870
9	81	3.000	729	2.080	**59**	3,481	7.681	205,379	3.893
10	100	3.162	1,000	2.154	**60**	3,600	7.745	216,000	3.914
11	121	3.316	1,331	2.223	**61**	3,721	7.810	226,981	3.936
12	144	3.464	1,728	2.289	**62**	3,844	7.874	238,328	3.957
13	169	3.605	2,197	2.351	**63**	3,969	7.937	250,047	3.979
14	196	3.741	2,744	2.410	**64**	4,096	8.000	262,144	4.000
15	225	3.872	3,375	2.466	**65**	4,225	8.062	274,625	4.020
16	256	4.000	4,096	2.519	**66**	4,356	8.124	287,496	4.041
17	289	4.123	4,913	2.571	**67**	4,489	8.185	300,763	4.061
18	324	4.242	5,832	2.620	**68**	4,624	8.246	314,432	4.081
19	361	4.358	6,859	2.668	**69**	4,761	8.306	328,509	4.101
20	400	4.472	8,000	2.714	**70**	4,900	8.366	343,000	4.121
21	441	4.582	9,261	2.758	**71**	5,041	8.426	357,911	4.140
22	484	4.690	10,648	2.802	**72**	5,184	8.485	373,248	4.160
23	529	4.795	12,167	2.843	**73**	5,329	8.544	389,017	4.179
24	576	4.898	13,824	2.884	**74**	5,476	8.602	405,224	4.198
25	625	5.000	15,625	2.924	**75**	5,625	8.660	421,875	4.217
26	676	5.099	17,576	2.962	**76**	5,776	8.717	438,976	4.235
27	729	5.196	19,683	3.000	**77**	5,929	8.774	456,533	4.254
28	784	5.291	21,952	3.036	**78**	6,084	8.831	474,552	4.272
29	841	5.385	24,389	3.072	**79**	6,241	8.888	493,039	4.290
30	900	5.477	27,000	3.107	**80**	6,400	8.944	512,000	4.308
31	961	5.567	29,791	3.141	**81**	6,561	9.000	531,441	4.326
32	1,024	5.656	32,768	3.174	**82**	6,724	9.055	551,368	4.344
33	1,089	5.744	35,937	3.207	**83**	6,889	9.110	571,787	4.362
34	1,156	5.830	39,304	3.239	**84**	7,056	9.165	592,704	4.379
35	1,225	5.916	42,875	3.271	**85**	7,225	9.219	614,125	4.396
36	1,296	6.000	46,656	3.301	**86**	7,396	9.273	636,056	4.414
37	1,369	6.082	50,653	3.332	**87**	7,569	9.327	658,503	4.431
38	1,444	6.164	54,872	3.361	**88**	7,744	9.380	681,472	4.447
39	1,521	6.245	59,319	3.391	**89**	7,921	9.433	704,969	4.464
40	1,600	6.324	64,000	3.419	**90**	8,100	9.486	729,000	4.481
41	1,681	6.403	68,921	3.448	**91**	8,281	9.539	753,571	4.497
42	1,764	6.480	74,088	3.476	**92**	8,464	9.591	778,688	4.514
43	1,849	6.557	79,507	3.503	**93**	8,649	9.643	804,357	4.530
44	1,936	6.633	85,184	3.530	**94**	8,836	9.695	830,584	4.546
45	2,025	6.708	91,125	3.556	**95**	9,025	9.746	857,375	4.562
46	2,116	6.782	97,336	3.583	**96**	9,216	9.797	884,736	4.578
47	2,209	6.855	103,823	3.608	**97**	9,409	9.848	912,673	4.594
48	2,304	6.928	110,592	3.634	**98**	9,604	9.899	941,192	4.610
49	2,401	7.000	117,649	3.659	**99**	9,801	9.949	970,299	4.626
50	2,500	7.071	125,000	3.684	**100**	10,000	10.000	1,000,000	4.641

TABLE II. Logarithms of Numbers

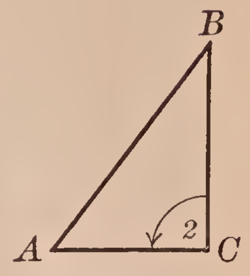
No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

TABLE II (*Continued*)

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

1. Recall the following definitions:

$$\sin A = \frac{BC}{AB}; \quad \cos A = \frac{AC}{AB}; \quad \tan A = \frac{BC}{AC}.$$



2. Observe the following new definition:

$$\text{cotangent } A = \frac{AC}{BC}, \text{ or } \cot A = \frac{AC}{BC}.$$

3. Observe that:

$$\sin B = \frac{AC}{AB}; \quad \cos B = \frac{BC}{AB}; \quad \tan B = \frac{AC}{BC}; \quad \cot B = \frac{BC}{AC}.$$

4.
$$\therefore \sin A = \frac{BC}{AB} = \cos B = \cos (90 - A).$$
$$\cos A = \frac{AC}{AB} = \sin B = \sin (90 - A).$$
$$\tan A = \frac{BC}{AC} = \cot B = \cot (90 - A).$$

In general, any function of an angle equals the co-function of the complementary angle.

5. This fact makes it possible to find the value of any function of an angle from 45° to 90° in a table of the values of the co-function of the angles from 0° to 45°.

Thus $\cos 84° = \sin 6°$; and $\sin 82° 30' = \cos 7° 30'$.

6. Table III is such a table. Observe its second column, headed by the abbreviation Sin. The numbers in this column are the sines of the angles opposite them at the left in column one. At the foot of column two is the abbreviation Cos. This indicates that the numbers in column two are also the cosines of certain angles. *They are the cosines of the angles opposite them at the right* in column six.

Thus: $\sin 6° = 0.1045 = \cos 84°$.

7. To find the value of a function of an angle greater than 45°, find the angle in column six; find the desired function *at the bottom of the page;* then, above that function, find the value which is opposite the given angle.

Thus $\cos 67° 40' = .3800$. (On page 259.)

TABLE III. Values of the Sines, Cosines, Tangents, and Cotangents of Certain Angles

Angle	Sin	Cos	Tan	Cot	Angle
0° 0′	.0000	1.0000	.0000		90° 0′
10′	.0029	1.0000	.0029	343.77	89° 50′
20′	.0058	1.0000	.0058	171.89	40′
30′	.0087	1.0000	.0087	114.59	30′
40′	.0116	.9999	.0116	85.940	20′
50′	.0145	.9999	.0145	68.750	89° 10′
1° 0′	.0175	.9998	.0175	57.290	89° 0′
10′	.0204	.9998	.0204	49.104	88° 50′
20′	.0233	.9997	.0233	42.964	40′
30′	.0262	.9997	.0262	38.188	30′
40′	.0291	.9996	.0291	34.368	20′
50′	.0320	.9995	.0320	31.242	88° 10′
2° 0′	.0349	.9994	.0349	28.636	88° 0′
10′	.0378	.9993	.0378	26.432	87° 50′
20′	.0407	.9992	.0407	24.542	40′
30′	.0436	.9990	.0437	22.904	30′
40′	.0465	.9989	.0466	21.470	20′
50′	.0494	.9988	.0495	20.206	87° 10′
3° 0′	.0523	.9986	.0524	19.081	87° 0′
10′	.0552	.9985	.0553	18.075	86° 50′
20′	.0581	.9983	.0582	17.169	40′
30′	.0610	.9981	.0612	16.350	30′
40′	.0640	.9980	.0641	15.605	20′
50′	.0669	.9978	.0670	14.924	86° 10′
4° 0′	.0698	.9976	.0699	14.301	86° 0′
10′	.0727	.9974	.0729	13.727	85° 50′
20′	.0756	.9971	.0758	13.197	40′
30′	.0785	.9969	.0787	12.706	30′
40′	.0814	.9967	.0816	12.251	20′
50′	.0843	.9964	.0846	11.826	85° 10′
5° 0′	.0872	.9962	.0875	11.430	85° 0′
10′	.0901	.9959	.0904	11.059	84° 50′
20′	.0929	.9957	.0934	10.712	40′
30′	.0958	.9954	.0963	10.385	30′
40′	.0987	.9951	.0992	10.078	20′
50′	.1016	.9948	.1022	9.7882	84° 10′
6° 0′	.1045	.9945	.1051	9.5144	84° 0′
10′	.1074	.9942	.1080	9.2553	83° 50′
20′	.1103	.9939	.1110	9.0098	40′
30′	.1132	.9936	.1139	8.7769	30′
40′	.1161	.9932	.1169	8.5555	20′
50′	.1190	.9929	.1198	8.3450	83° 10′
7° 0′	.1219	.9925	.1228	8.1443	83° 0′
10′	.1248	.9922	.1257	7.9530	82° 50′
20′	.1276	.9918	.1287	7.7704	40′
30′	.1305	.9914	.1317	7.5958	30′
40′	.1334	.9911	.1346	7.4287	20′
50′	.1363	.9907	.1376	7.2687	82° 10′
8° 0′	.1392	.9903	.1405	7.1154	82° 0′
10′	.1421	.9899	.1435	6.9682	81° 50′
20′	.1449	.9894	.1465	6.8269	40′
30′	.1478	.9890	.1495	6.6912	30′
40′	.1507	.9886	.1524	6.5606	20′
50′	.1536	.9881	.1554	6.4348	81° 10′
9° 0′	.1564	.9877	.1584	6.3138	81° 0′
Angle	Cos	Sin	Cot	Tan	Angle

TABLE III (*Continued*)

Angle	Sin	Cos	Tan	Cot	Angle
9° 0′	.1564	.9877	.1584	6.3138	**81° 0′**
10′	.1593	.9872	.1614	6.1970	**80° 50′**
20′	.1622	.9868	.1644	6.0844	**40′**
30′	.1650	.9863	.1673	5.9758	**30′**
40′	.1679	.9858	.1703	5.8708	**20′**
50′	.1708	.9853	.1733	5.7694	**80° 10′**
10° 0′	.1736	.9848	.1763	5.6713	**80° 0′**
10′	.1765	.9843	.1793	5.5764	**79° 50′**
20′	.1794	.9838	.1823	5.4845	**40′**
30′	.1822	.9833	.1853	5.3955	**30′**
40′	.1851	.9827	.1883	5.3093	**20′**
50′	.1880	.9822	.1914	5.2257	**79° 10′**
11° 0′	.1908	.9816	.1944	5.1446	**79° 0′**
10′	.1937	.9811	.1974	5.0658	**78° 50′**
20′	.1965	.9805	.2004	4.9894	**40′**
30′	.1994	.9799	.2035	4.9152	**30′**
40′	.2022	.9793	.2065	4.8430	**20′**
50′	.2051	.9787	.2095	4.7729	**78° 10′**
12° 0′	.2079	.9781	.2126	4.7046	**78° 0′**
10′	.2108	.9775	.2156	4.6382	**77° 50′**
20′	.2136	.9769	.2186	4.5736	**40′**
30′	.2164	.9763	.2217	4.5107	**30′**
40′	.2193	.9757	.2247	4.4494	**20′**
50′	.2221	.9750	.2278	4.3897	**77° 10′**
13° 0′	.2250	.9744	.2309	4.3315	**77° 0′**
10′	.2278	.9737	.2339	4.2747	**76° 50′**
20′	.2306	.9730	.2370	4.2193	**40′**
30′	.2334	.9724	.2401	4.1653	**30′**
40′	.2363	.9717	.2432	4.1126	**20′**
50′	.2391	.9710	.2462	4.0611	**76° 10′**
14° 0′	.2419	.9703	.2493	4.0108	**76° 0′**
10′	.2447	.9696	.2524	3.9617	**75° 50′**
20′	.2476	.9689	.2555	3.9136	**40′**
30′	.2504	.9681	.2586	3.8667	**30′**
40′	.2532	.9674	.2617	3.8208	**20′**
50′	.2560	.9667	.2648	3.7760	**75° 10′**
15° 0′	.2588	.9659	.2679	3.7321	**75° 0′**
10′	.2616	.9652	.2711	3.6891	**74° 50′**
20′	.2644	.9644	.2742	3.6470	**40′**
30′	.2672	.9636	.2773	3.6059	**30′**
40′	.2700	.9628	.2805	3.5656	**20′**
50′	.2728	.9621	.2836	3.5261	**74° 10′**
16° 0′	.2756	.9613	.2867	3.4874	**74° 0′**
10′	.2784	.9605	.2899	3.4495	**73° 50′**
20′	.2812	.9596	.2931	3.4124	**40′**
30′	.2840	.9588	.2962	3.3759	**30′**
40′	.2868	.9580	.2994	3.3402	**20′**
50′	.2896	.9572	.3026	3.3052	**73° 10′**
17° 0′	.2924	.9563	.3057	3.2709	**73° 0′**
10′	.2952	.9555	.3089	3.2371	**72° 50′**
20′	.2979	.9546	.3121	3.2041	**40′**
30′	.3007	.9537	.3153	3.1716	**30′**
40′	.3035	.9528	.3185	3.1397	**20′**
50′	.3062	.9520	.3217	3.1084	**72° 10′**
18° 0′	.3090	.9511	.3249	3.0777	**72° 0′**
Angle	Cos	Sin	Cot	Tan	Angle

TABLE III (*Continued*)

Angle	Sin	Cos	Tan	Cot	Angle
18° 0′	.3090	.9511	.3249	3.0777	**72° 0′**
10′	.3118	.9502	.3281	3.0475	**71° 50′**
20′	.3145	.9492	.3314	3.0178	**40′**
30′	.3173	.9483	.3346	2.9887	**30′**
40′	.3201	.9474	.3378	2.9600	**20′**
50′	.3228	.9465	.3411	2.9319	**71° 10′**
19° 0′	.3256	.9455	.3443	2.9042	**71° 0′**
10′	.3283	.9446	.3476	2.8770	**70° 50′**
20′	.3311	.9436	.3508	2.8502	**40′**
30′	.3338	.9426	.3541	2.8239	**30′**
40′	.3365	.9417	.3574	2.7980	**20′**
50′	.3393	.9407	.3607	2.7725	**70° 10′**
20° 0′	.3420	.9397	.3640	2.7475	**70° 0′**
10′	.3448	.9387	.3673	2.7228	**69° 50′**
20′	.3475	.9377	.3706	2.6985	**40′**
30′	.3502	.9367	.3739	2.6746	**30′**
40′	.3529	.9356	.3772	2.6511	**20′**
50′	.3557	.9346	.3805	2.6279	**69° 10′**
21° 0′	.3584	.9336	.3839	2.6051	**69° 0′**
10′	.3611	.9325	.3872	2.5826	**68° 50′**
20′	.3638	.9315	.3906	2.5605	**40′**
30′	.3665	.9304	.3939	2.5386	**30′**
40′	.3692	.9293	.3973	2.5172	**20′**
50′	.3719	.9283	.4006	2.4960	**68° 10′**
22° 0′	.3746	.9272	.4040	2.4751	**68° 0′**
10′	.3773	.9261	.4074	2.4545	**67° 50′**
20′	.3800	.9250	.4108	2.4342	**40′**
30′	.3827	.9239	.4142	2.4142	**30′**
40′	.3854	.9228	.4176	2.3945	**20′**
50′	.3881	.9216	.4210	2.3750	**67° 10′**
23° 0′	.3907	.9205	.4245	2.3559	**67° 0′**
10′	.3934	.9194	.4279	2.3369	**66° 50′**
20′	.3961	.9182	.4314	2.3183	**40′**
30′	.3987	.9171	.4348	2.2998	**30′**
40′	.4014	.9159	.4383	2.2817	**20′**
50′	.4041	.9147	.4417	2.2637	**66° 10′**
24° 0′	.4067	.9135	.4452	2.2460	**66° 0′**
10′	.4094	.9124	.4487	2.2286	**65° 50′**
20′	.4120	.9112	.4522	2.2113	**40′**
30′	.4147	.9100	.4557	2.1943	**30′**
40′	.4173	.9088	.4592	2.1775	**20′**
50′	.4200	.9075	.4628	2.1609	**65° 10′**
25° 0′	.4226	.9063	.4663	2.1445	**65° 0′**
10′	.4253	.9051	.4699	2.1283	**64° 50′**
20′	.4279	.9038	.4734	2.1123	**40′**
30′	.4305	.9026	.4770	2.0965	**30′**
40′	.4331	.9013	.4806	2.0809	**20′**
50′	.4358	.9001	.4841	2.0655	**64° 10′**
26° 0′	.4384	.8988	.4877	2.0503	**64° 0′**
10′	.4410	.8975	.4913	2.0353	**63° 50′**
20′	.4436	.8962	.4950	2.0204	**40′**
30′	.4462	.8949	.4986	2.0057	**30′**
40′	.4488	.8936	.5022	1.9912	**20′**
50′	.4514	.8923	.5059	1.9768	**63° 10′**
27° 0′	.4540	.8910	.5095	1.9626	**63° 0′**
Angle	Cos	Sin	Cot	Tan	Angle

TABLE III (*Continued*)

Angle	Sin	Cos	Tan	Cot	Angle
27° 0′	.4540	.8910	.5095	1.9626	**63° 0′**
10′	.4566	.8897	.5132	1.9486	**62° 50′**
20′	.4592	.8884	.5169	1.9347	**40′**
30′	.4617	.8870	.5206	1.9210	**30′**
40′	.4643	.8857	.5243	1.9074	**20′**
50′	.4669	.8843	.5280	1.8940	**62° 10′**
28° 0′	.4695	.8829	.5317	1.8807	**62° 0′**
10′	.4720	.8816	.5354	1.8676	**61° 50′**
20′	.4746	.8802	.5392	1.8546	**40′**
30′	.4772	.8788	.5430	1.8418	**30′**
40′	.4797	.8774	.5467	1.8291	**20′**
50′	.4823	.8760	.5505	1.8165	**61° 10′**
29° 0′	.4848	.8746	.5543	1.8040	**61° 0′**
10′	.4874	.8732	.5581	1.7917	**60° 50′**
20′	.4899	.8718	.5619	1.7796	**40′**
30′	.4924	.8704	.5658	1.7675	**30′**
40′	.4950	.8689	.5696	1.7556	**20′**
50′	.4975	.8675	.5735	1.7437	**60° 10′**
30° 0′	.5000	.8660	.5774	1.7321	**60° 0′**
10′	.5025	.8646	.5812	1.7205	**59° 50′**
20′	.5050	.8631	.5851	1.7090	**40′**
30′	.5075	.8616	.5890	1.6977	**30′**
40′	.5100	.8601	.5930	1.6864	**20′**
50′	.5125	.8587	.5969	1.6753	**59° 10′**
31° 0′	.5150	.8572	.6009	1.6643	**59° 0′**
10′	.5175	.8557	.6048	1.6534	**58° 50′**
20′	.5200	.8542	.6088	1.6426	**40′**
30′	.5225	.8526	.6128	1.6319	**30′**
40′	.5250	.8511	.6168	1.6212	**20′**
50′	.5275	.8496	.6208	1.6107	**58° 10′**
32° 0′	.5299	.8480	.6249	1.6003	**58° 0′**
10′	.5324	.8465	.6289	1.5900	**57° 50′**
20′	.5348	.8450	.6330	1.5798	**40′**
30′	.5373	.8434	.6371	1.5697	**30′**
40′	.5398	.8418	.6412	1.5597	**20′**
50′	.5422	.8403	.6453	1.5497	**57° 10′**
33° 0′	.5446	.8387	.6494	1.5399	**57° 0′**
10′	.5471	.8371	.6536	1.5301	**56° 50′**
20′	.5495	.8355	.6577	1.5204	**40′**
30′	.5519	.8339	.6619	1.5108	**30′**
40′	.5544	.8323	.6661	1.5013	**20′**
50′	.5568	.8307	.6703	1.4919	**56° 10′**
34° 0′	.5592	.8290	.6745	1.4826	**56° 0′**
10′	.5616	.8274	.6787	1.4733	**55° 50′**
20′	.5640	.8258	.6830	1.4641	**40′**
30′	.5664	.8241	.6873	1.4550	**30′**
40′	.5688	.8225	.6916	1.4460	**20′**
50′	.5712	.8208	.6959	1.4370	**55° 10′**
35° 0′	.5736	.8192	.7002	1.4281	**55° 0′**
10′	.5760	.8175	.7046	1.4193	**54° 50′**
20′	.5783	.8158	.7089	1.4106	**40′**
30′	.5807	.8141	.7133	1.4019	**30′**
40′	.5831	.8124	.7177	1.3934	**20′**
50′	.5854	.8107	.7221	1.3848	**54° 10′**
36° 0′	.5878	.8090	.7265	1.3764	**54° 0′**
Angle	Cos	Sin	Cot	Tan	Angle

TABLE III (*Continued*)

Angle	Sin	Cos	Tan	Cot	Angle
36° 0′	.5878	.8090	.7265	1.3764	**54° 0′**
10′	.5901	.8073	.7310	1.3680	**53° 50′**
20′	.5925	.8056	.7355	1.3597	**40′**
30′	.5948	.8039	.7400	1.3514	**30′**
40′	.5972	.8021	.7445	1.3432	**20′**
50′	.5995	.8004	.7490	1.3351	**53° 10′**
37° 0′	.6018	.7986	.7536	1.3270	**53° 0′**
10′	.6041	.7969	.7581	1.3190	**52° 50′**
20′	.6065	.7951	.7627	1.3111	**40′**
30′	.6088	.7934	.7673	1.3032	**30′**
40′	.6111	.7916	.7720	1.2954	**20′**
50′	.6134	.7898	.7766	1.2876	**52° 10′**
38° 0′	.6157	.7880	.7813	1.2799	**52° 0′**
10′	.6180	.7862	.7860	1.2723	**51° 50′**
20′	.6202	.7844	.7907	1.2647	**40′**
30′	.6225	.7826	.7954	1.2572	**30′**
40′	.6248	.7808	.8002	1.2497	**20′**
50′	.6271	.7790	.8050	1.2423	**51° 10′**
39° 0′	.6293	.7771	.8098	1.2349	**51° 0′**
10′	.6316	.7753	.8146	1.2276	**50° 50′**
20′	.6338	.7735	.8195	1.2203	**40′**
30′	.6361	.7716	.8243	1.2131	**30′**
40′	.6383	.7698	.8292	1.2059	**20′**
50′	.6406	.7679	.8342	1.1988	**50° 10′**
40° 0′	.6428	.7660	.8391	1.1918	**50° 0′**
10′	.6450	.7642	.8441	1.1847	**49° 50′**
20′	.6472	.7623	.8491	1.1778	**40′**
30′	.6494	.7604	.8541	1.1708	**30′**
40′	.6517	.7585	.8591	1.1640	**20′**
50′	.6539	.7566	.8642	1.1571	**49° 10′**
41° 0′	.6561	.7547	.8693	1.1504	**49° 0′**
10′	.6583	.7528	.8744	1.1436	**48° 50′**
20′	.6604	.7509	.8796	1.1369	**40′**
30′	.6626	.7490	.8847	1.1303	**30′**
40′	.6648	.7470	.8899	1.1237	**20′**
50′	.6670	.7451	.8952	1.1171	**48° 10′**
42° 0′	.6691	.7431	.9004	1.1106	**48° 0′**
10′	.6713	.7412	.9057	1.1041	**47° 50′**
20′	.6734	.7392	.9110	1.0977	**40′**
30′	.6756	.7373	.9163	1.0913	**30′**
40′	.6777	.7353	.9217	1.0850	**20′**
50′	.6799	.7333	.9271	1.0786	**47° 10′**
43° 0′	.6820	.7314	.9325	1.0724	**47° 0′**
10′	.6841	.7294	.9380	1.0661	**46° 50′**
20′	.6862	.7274	.9435	1.0599	**40′**
30′	.6884	.7254	.9490	1.0538	**30′**
40′	.6905	.7234	.9545	1.0477	**20′**
50′	.6926	.7214	.9601	1.0416	**46° 10′**
44° 0′	.6947	.7193	.9657	1.0355	**46° 0′**
10′	.6967	.7173	.9713	1.0295	**45° 50′**
20′	.6988	.7153	.9770	1.0235	**40′**
30′	.7009	.7133	.9827	1.0176	**30′**
40′	.7030	.7112	.9884	1.0117	**20′**
50′	.7050	.7092	.9942	1.0058	**45° 10′**
45° 0′	.7071	.7071	1.0000	1.0000	**45° 0′**
Angle	Cos	Sin	Cot	Tan	Angle

TABLE IV. LOGARITHMS OF THE SINES, COSINES, TANGENTS, AND COTANGENTS OF CERTAIN ANGLES (INCREASED BY 10)

ANGLE	LOG SIN	LOG COS	LOG TAN	LOG COT	ANGLE
0° 0′					**90° 0′**
10′	7.4637	10.0000	7.4637	12.5363	**89° 50′**
20′	.7648	.0000	.7648	.2352	**40′**
30′	.9408	.0000	.9409	.0591	**30′**
40′	8.0658	.0000	8.0658	11.9342	**20′**
50′	.1627	.0000	.1627	.8373	**89° 10′**
1° 0′	8.2419	9.9999	8.2419	11.7581	**89° 0′**
10′	.3088	.9999	.3089	.6911	**88° 50′**
20′	.3668	.9999	.3669	.6331	**40′**
30′	.4179	.9999	.4181	.5819	**30′**
40′	.4637	.9998	.4638	.5362	**20′**
50′	.5050	.9998	.5053	.4947	**88° 10′**
2° 0′	8.5428	9.9997	8.5431	11.4569	**88° 0′**
10′	.5776	.9997	.5779	.4221	**87° 50′**
20′	.6097	.9996	.6101	.3899	**40′**
30′	.6397	.9996	.6401	.3599	**30′**
40′	.6677	.9995	.6682	.3318	**20′**
50′	.6940	.9995	.6945	.3055	**87° 10′**
3° 0′	8.7188	9.9994	8.7194	11.2806	**87° 0′**
10′	.7423	.9993	.7429	.2571	**86° 50′**
20′	.7645	.9993	.7652	.2348	**40′**
30′	.7857	.9992	.7865	.2135	**30′**
40′	.8059	.9991	.8067	.1933	**20′**
50′	.8251	.9990	.8261	.1739	**86° 10′**
4° 0′	8.8436	9.9989	8.8446	11.1554	**86° 0′**
10′	.8613	.9989	.8624	.1376	**85° 50′**
20′	.8783	.9988	.8795	.1205	**40′**
30′	.8946	.9987	.8960	.1040	**30′**
40′	.9104	.9986	.9118	.0882	**20′**
50′	.9256	.9985	.9272	.0728	**85° 10′**
5° 0′	8.9403	9.9983	8.9420	11.0580	**85° 0′**
10′	.9545	.9982	.9563	.0437	**84° 50′**
20′	.9682	.9981	.9701	.0299	**40′**
30′	.9816	.9980	.9836	.0164	**30′**
40′	.9945	.9979	.9966	.0034	**20′**
50′	9.0070	.9977	9.0093	10.9907	**84° 10′**
6° 0′	9.0192	9.9976	9.0216	.9784	**84° 0′**
10′	.0311	.9975	.0336	10.9664	**83° 50′**
20′	.0426	.9973	.0453	.9547	**40′**
30′	.0539	.9972	.0567	.9433	**30′**
40′	.0648	.9971	.0678	.9322	**20′**
50′	.0755	.9969	.0786	.9214	**83° 10′**
7° 0′	9.0859	9.9968	9.0891	10.9109	**83° 0′**
10′	.0961	.9966	.0995	.9005	**82° 50′**
20′	.1060	.9964	.1096	.8904	**40′**
30′	.1157	.9963	.1194	.8806	**30′**
40′	.1252	.9961	.1291	.8709	**20′**
50′	.1345	.9959	.1385	.8615	**82° 10′**
8° 0′	9.1436	9.9958	9.1478	10.8522	**82° 0′**
10′	.1525	.9956	.1569	.8431	**81° 50′**
20′	.1612	.9954	.1658	.8342	**40′**
30′	.1697	.9952	.1745	.8255	**30′**
40′	.1781	.9950	.1831	.8169	**20′**
50′	.1863	.9948	.1915	.8085	**81° 10′**
9° 0′	9.1943	9.9946	9.1997	10.8003	**81° 0′**
ANGLE	LOG COS	LOG SIN	LOG COT	LOG TAN	ANGLE

TABLE IV (*Continued*)

Angle	Log sin	Log cos	Log tan	Log cot	Angle
9° 0′	9.1943	9.9946	9.1997	10.8003	**81° 0′**
10′	.2022	.9944	.2078	.7922	**80° 50′**
20′	.2100	.9942	.2158	.7842	**40′**
30′	.2176	.9940	.2236	.7764	**30′**
40′	.2251	.9938	.2313	.7687	**20′**
50′	.2324	.9936	.2389	.7611	**80° 10′**
10° 0′	9.2397	9.9934	9.2463	10.7537	**80° 0′**
10′	.2468	.9931	.2536	.7464	**79° 50′**
20′	.2538	.9929	.2609	.7391	**40′**
30′	.2606	.9927	.2680	.7320	**30′**
40′	.2674	.9924	.2750	.7250	**20′**
50′	.2740	.9922	.2819	.7181	**79° 10′**
11° 0′	9.2806	9.9919	9.2887	10.7113	**79° 0′**
10′	.2870	.9917	.2953	.7047	**78° 50′**
20′	.2934	.9914	.3020	.6980	**40′**
30′	.2997	.9912	.3085	.6915	**30′**
40′	.3058	.9909	.3149	.6851	**20′**
50′	.3119	.9907	.3212	.6788	**78° 10′**
12° 0′	9.3179	9.9904	9.3275	10.6725	**78° 0′**
10′	.3238	.9901	.3336	.6664	**77° 50′**
20′	.3296	.9899	.3397	.6603	**40′**
30′	.3353	.9896	.3458	.6542	**30′**
40′	.3410	.9893	.3517	.6483	**20′**
50′	.3466	.9890	.3576	.6424	**77° 10′**
13° 0′	9.3521	9.9887	9.3634	10.6366	**77° 0′**
10′	.3575	.9884	.3691	.6309	**76° 50′**
20′	.3629	.9881	.3748	.6252	**40′**
30′	.3682	.9878	.3804	.6196	**30′**
40′	.3734	.9875	.3859	.6141	**20′**
50′	.3786	.9872	.3914	.6086	**76° 10′**
14° 0′	9.3837	9.9869	9.3968	10.6032	**76° 0′**
10′	.3887	.9866	.4021	.5979	**75° 50′**
20′	.3937	.9863	.4074	.5926	**40′**
30′	.3986	.9859	.4127	.5873	**30′**
40′	.4035	.9856	.4178	.5822	**20′**
50′	.4083	.9853	.4230	.5770	**75° 10′**
15° 0′	9.4130	9.9849	9.4281	10.5719	**75° 0′**
10′	.4177	.9846	.4331	.5669	**74° 50′**
20′	.4223	.9843	.4381	.5619	**40′**
30′	.4269	.9839	.4430	.5570	**30′**
40′	.4314	.9836	.4479	.5521	**20′**
50′	.4359	.9832	.4527	.5473	**74° 10′**
16° 0′	9.4403	9.9828	9.4575	10.5425	**74° 0′**
10′	.4447	.9825	.4622	.5378	**73° 50′**
20′	.4491	.9821	.4669	.5331	**40′**
30′	.4533	.9817	.4716	.5284	**30′**
40′	.4576	.9814	.4762	.5238	**20′**
50′	.4618	.9810	.4808	.5192	**73° 10′**
17° 0′	9.4659	9.9806	9.4853	10.5147	**73° 0′**
10′	.4700	.9802	.4898	.5102	**72° 50′**
20′	.4741	.9798	.4943	.5057	**40′**
30′	.4781	.9794	.4987	.5013	**30′**
40′	.4821	.9790	.5031	.4969	**20′**
50′	.4861	.9786	.5075	.4925	**72° 10′**
18° 0′	9.4900	9.9782	9.5118	10.4882	**72° 0′**
Angle	Log cos	Log sin	Log cot	Log tan	Angle

TABLE IV (*Continued*)

Angle	Log sin	Log cos	Log tan	Log cot	Angle
18° 0′	9.4900	9.9782	9.5118	10.4882	**72° 0′**
10′	.4939	.9778	.5161	.4839	**71° 50′**
20′	.4977	.9774	.5203	.4797	**40′**
30′	.5015	.9770	.5245	.4755	**30′**
40′	.5052	.9765	.5287	.4713	**20′**
50′	.5090	.9761	.5329	.4671	**71° 10′**
19° 0′	9.5126	9.9757	9.5370	10.4630	**71° 0′**
10′	.5163	.9752	.5411	.4589	**70° 50′**
20′	.5199	.9748	.5451	.4549	**40′**
30′	.5235	.9743	.5491	.4509	**30′**
40′	.5270	.9739	.5531	.4469	**20′**
50′	.5306	.9734	.5571	.4429	**70° 10′**
20° 0′	9.5341	9.9730	9.5611	10.4389	**70° 0′**
10′	.5375	.9725	.5650	.4350	**69° 50′**
20′	.5409	.9721	.5689	.4311	**40′**
30′	.5443	.9716	.5727	.4273	**30′**
40′	.5477	.9711	.5766	.4234	**20′**
50′	.5510	.9706	.5804	.4196	**69° 10′**
21° 0′	9.5543	9.9702	9.5842	10.4158	**69° 0′**
10′	.5576	.9697	.5879	.4121	**68° 50′**
20′	.5609	.9692	.5917	.4083	**40′**
30′	.5641	.9687	.5954	.4046	**30′**
40′	.5673	.9682	.5991	.4009	**20′**
50′	.5704	.9677	.6028	.3972	**68° 10′**
22° 0′	9.5736	9.9672	9.6064	10.3936	**68° 0′**
10′	.5767	.9667	.6100	.3900	**67° 50′**
20′	.5798	.9661	.6136	.3864	**40′**
30′	.5828	.9656	.6172	.3828	**30′**
40′	.5859	.9651	.6208	.3792	**20′**
50′	.5889	.9646	.6243	.3757	**67° 10′**
23° 0′	9.5919	9.9640	9.6279	10.3721	**67° 0′**
10′	.5948	.9635	.6314	.3686	**66° 50′**
20′	.5978	.9629	.6348	.3652	**40′**
30′	.6007	.9624	.6383	.3617	**30′**
40′	.6036	.9618	.6417	.3583	**20′**
50′	.6065	.9613	.6452	.3548	**66° 10′**
24° 0′	9.6093	9.9607	9.6486	10.3514	**66° 0′**
10′	.6121	.9602	.6520	.3480	**65° 50′**
20′	.6149	.9596	.6553	.3447	**40′**
30′	.6177	.9590	.6587	.3413	**30′**
40′	.6205	.9584	.6620	.3380	**20′**
50′	.6232	.9579	.6654	.3346	**65° 10′**
25° 0′	9.6259	9.9573	9.6687	10.3313	**65° 0′**
10′	.6286	.9567	.6720	.3280	**64° 50′**
20′	.6313	.9561	.6752	.3248	**40′**
30′	.6340	.9555	.6785	.3215	**30′**
40′	.6366	.9549	.6817	.3183	**20′**
50′	.6392	.9543	.6850	.3150	**64° 10′**
26° 0′	9.6418	9.9537	9.6882	10.3118	**64° 0′**
10′	.6444	.9530	.6914	.3086	**63° 50′**
20′	.6470	.9524	.6946	.3054	**40′**
30′	.6495	.9518	.6977	.3023	**30′**
40′	.6521	.9512	.7009	.2991	**20′**
50′	.6546	.9505	.7040	.2960	**63° 10′**
27° 0′	9.6570	9.9499	9.7072	10.2928	**63° 0′**
Angle	Log cos	Log sin	Log cot	Log tan	Angle

TABLE IV (*Continued*)

Angle	Log sin	Log cos	Log tan	Log cot	Angle
27° 0′	9.6570	9.9499	9.7072	10.2928	**63° 0′**
10′	.6595	.9492	.7103	.2897	**62° 50′**
20′	.6620	.9486	.7134	.2866	**40′**
30′	.6644	.9479	.7165	.2835	**30′**
40′	.6668	.9473	.7196	.2804	**20′**
50′	.6692	.9466	.7226	.2774	**62° 10′**
28° 0′	9.6716	9.9459	9.7257	10.2743	**62° 0′**
10′	.6740	.9453	.7287	.2713	**61° 50′**
20′	.6763	.9446	.7317	.2683	**40′**
30′	.6787	.9439	.7348	.2652	**30′**
40′	.6810	.9432	.7378	.2622	**20′**
50′	.6833	.9425	.7408	.2592	**61° 10′**
29° 0′	9.6856	9.9418	9.7438	10.2562	**61° 0′**
10′	.6878	.9411	.7467	.2533	**60° 50′**
20′	.6901	.9404	.7497	.2503	**40′**
30′	.6923	.9397	.7526	.2474	**30′**
40′	.6946	.9390	.7556	.2444	**20′**
50′	.6968	.9383	.7585	.2415	**60° 10′**
30° 0′	9.6990	9.9375	9.7614	10.2386	**60° 0′**
10′	.7012	.9368	.7644	.2356	**59° 50′**
20′	.7033	.9361	.7673	.2327	**40′**
30′	.7055	.9353	.7701	.2299	**30′**
40′	.7076	.9346	.7730	.2270	**20′**
50′	.7097	.9338	.7759	.2241	**59° 10′**
31° 0′	9.7118	9.9331	9.7788	10.2212	**59° 0′**
10′	.7139	.9323	.7816	.2184	**58° 50′**
20′	.7160	.9315	.7845	.2155	**40′**
30′	.7181	.9308	.7873	.2127	**30′**
40′	.7201	.9300	.7902	.2098	**20′**
50′	.7222	.9292	.7930	.2070	**58° 10′**
32° 0′	9.7242	9.9284	9.7958	10.2042	**58° 0′**
10′	.7262	.9276	.7986	.2014	**57° 50′**
20′	.7282	.9268	.8014	.1986	**40′**
30′	.7302	.9260	.8042	.1958	**30′**
40′	.7322	.9252	.8070	.1930	**20′**
50′	.7342	.9244	.8097	.1903	**57° 10′**
33° 0′	9.7361	9.9236	9.8125	10.1875	**57° 0′**
10′	.7380	.9228	.8153	.1847	**56° 50′**
20′	.7400	.9219	.8180	.1820	**40′**
30′	.7419	.9211	.8208	.1792	**30′**
40′	.7438	.9203	.8235	.1765	**20′**
50′	.7457	.9194	.8263	.1737	**56° 10′**
34° 0′	9.7476	9.9186	9.8290	10.1710	**56° 0′**
10′	.7494	.9177	.8317	.1683	**55° 50′**
20′	.7513	.9169	.8344	.1656	**40′**
30′	.7531	.9160	.8371	.1629	**30′**
40′	.7550	.9151	.8398	.1602	**20′**
50′	.7568	.9142	.8425	.1575	**55° 10′**
35° 0′	9.7586	9.9134	9.8452	10.1548	**55° 0′**
10′	.7604	.9125	.8479	.1521	**54° 50′**
20′	.7622	.9116	.8506	.1494	**40′**
30′	.7640	.9107	.8533	.1467	**30′**
40′	.7657	.9098	.8559	.1441	**20′**
50′	.7675	.9089	.8586	.1414	**54° 10′**
36° 0′	9.7692	9.9080	9.8613	10.1387	**54° 0′**
Angle	Log cos	Log sin	Log cot	Log tan	Angle

TABLE IV (*Continued*)

Angle	Log sin	Log cos	Log tan	Log cot	Angle
36° 0′	9.7692	9.9080	9.8613	10.1387	**54° 0′**
10′	.7710	.9070	.8639	.1361	**53° 50′**
20′	.7727	.9061	.8666	.1334	**40′**
30′	.7744	.9052	.8692	.1308	**30′**
40′	.7761	.9042	.8718	.1282	**20′**
50′	.7778	.9033	.8745	.1255	**53° 10′**
37° 0′	9.7795	9.9023	9.8771	10.1229	**53° 0′**
10′	.7811	.9014	.8797	.1203	**52° 50′**
20′	.7828	.9004	.8824	.1176	**40′**
30′	.7844	.8995	.8850	.1150	**30′**
40′	.7861	.8985	.8876	.1124	**20′**
50′	.7877	.8975	.8902	.1098	**52° 10′**
38° 0′	9.7893	9.8965	9.8928	10.1072	**52° 0′**
10′	.7910	.8955	.8954	.1046	**51° 50′**
20′	.7926	.8945	.8980	.1020	**40′**
30′	.7941	.8935	.9006	.0994	**30′**
40′	.7957	.8925	.9032	.0968	**20′**
50′	.7973	.8915	.9058	.0942	**51° 10′**
39° 0′	9.7989	9.8905	9.9084	10.0916	**51° 0′**
10′	.8004	.8895	.9110	.0890	**50° 50′**
20′	.8020	.8884	.9135	.0865	**40′**
30′	.8035	.8874	.9161	.0839	**30′**
40′	.8050	.8864	.9187	.0813	**20′**
50′	.8066	.8853	.9212	.0788	**50° 10′**
40° 0′	9.8081	9.8843	9.9238	10.0762	**50° 0′**
10′	.8096	.8832	.9264	.0736	**49° 50′**
20′	.8111	.8821	.9289	.0711	**40′**
30′	.8125	.8810	.9315	.0685	**30′**
40′	.8140	.8800	.9341	.0659	**20′**
50′	.8155	.8789	.9366	.0634	**49° 10′**
41° 0′	9.8169	9.8778	9.9392	10.0608	**49° 0′**
10′	.8184	.8767	.9417	.0583	**48° 50′**
20′	.8198	.8756	.9443	.0557	**40′**
30′	.8213	.8745	.9468	.0532	**30′**
40′	.8227	.8733	.9494	.0506	**20′**
50′	.8241	.8722	.9519	.0481	**48° 10′**
42° 0′	9.8255	9.8711	9.9544	10.0456	**48° 0′**
10′	.8269	.8699	.9570	.0430	**47° 50′**
20′	.8283	.8688	.9595	.0405	**40′**
30′	.8297	.8676	.9621	.0379	**30′**
40′	.8311	.8665	.9646	.0354	**20′**
50′	.8324	.8653	.9671	.0329	**47° 10′**
43° 0′	9.8338	9.8641	9.9697	10.0303	**47° 0′**
10′	.8351	.8629	.9722	.0278	**46° 50′**
20′	.8365	.8618	.9747	.0253	**40′**
30′	.8378	.8606	.9773	.0228	**30′**
40′	.8391	.8594	.9798	.0202	**20′**
50′	.8405	.8582	.9823	.0177	**46° 10′**
44° 0′	9.8418	9.8569	9.9848	10.0152	**46° 0′**
10′	.8431	.8557	.9874	.0126	**45° 50′**
20′	.8444	.8545	.9899	.0101	**40′**
30′	.8457	.8532	.9924	.0076	**30′**
40′	.8469	.8520	.9949	.0051	**20′**
50′	.8482	.8507	.9975	.0025	**45° 10′**
45° 0′	9.8495	9.8495	10.0000	10.0000	**45° 0′**
Angle	Log cos	Log sin	Log cot	Log tan	Angle